Global exponential stability and periodic solutions of recurrent neural networks with delays

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Abstract

In this Letter, by utilizing the Lyapunov functional method, applying M-matrix and topological degree theory, we analyze the global exponential stability and the existence of periodic solutions of a class of recurrent neural networks with delays. Some simple and new sufficient conditions ensuring existence, uniqueness and global exponential stability of the equilibrium point and periodic solutions of delayed recurrent neural networks are obtained, which do not require the activation functions to be differentiable, bounded and monotone nondecreasing. In addition, two examples are also given to illustrate the theory. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The dynamics of recurrent neural networks such as cellular neural networks (CNNs) and delayed cellular neural networks (DCNNs) have been deeply investigated in recent years due to its applicability in solving some image processing, signal processing and pattern recognition problems. Several important results have been obtained in Refs. [1–25,29–33].

It is well-known that the models of DCNNs are described by the following system of differential equations

\[ x'_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_j)) + I_i, \quad i = 1, 2, \ldots, n, \tag{1} \]

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where $c_i > 0$, $a_{ij}$, $b_{ij}$, $I_i$ are constants, $\tau_j$ is not negative constant, $f_j$ is activation function.

For system (1), there exist some results on global asymptotical stability (GAS), global exponential stability (GES) and periodic solutions. For different aspects of the stability theory of CNNs and DCNNs, the interested reader is referred to Refs. [3,6,9–12] and Refs. [4,5,8,13–16,20,21,24,25], respectively, and the references cited therein. In addition, in [22,23] the authors have discussed the absolute stability of delayed neural networks.

In this Letter, we consider the global exponential stability (GES) and periodic solutions of a class of delayed recurrent neural networks with general activation functions, which described by the differential equations

$$x_i'(t) = -c_i h_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij})) + I_i(t), \quad i = 1, 2, \ldots, n,$$

(2)

where $c_i > 0$, $n$ denotes the number of units in a neural network, $x_i(t)$ corresponds to the state of the $i$th unit at time $t$, $f_j(x_j(t))$, $g_j(x_j(t))$ denote the activation functions of the $j$th unit at time $t$, $a_{ij}$, $b_{ij}$, $c_i$ are constants, $\alpha_j$ denotes the strength of the $j$th unit at time $t$, $b_{ij}$ denotes the strength of the $j$th unit on the $i$th unit at time $t - \tau_{ij}$, $I_i(t)$ is the external bias on the $i$th unit at time $t$, $\gamma_i$ is nonnegative constant, $c_i$ represents the rate with which the $i$th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs.

Suppose further that the following assumptions are satisfied

(H1): $h_i : R \to R$ is differentiable and $\gamma_i = \inf_{x \in R} h_i'(x) > 0$, $h_i(0) = 0$, where $h_i'(x)$ represents the derivative of $h_i(x)$.

(H2): There are some number $\alpha_i > 0$, $\beta_i > 0$ such that

$$|f_i(x_1) - f_i(x_2)| \leq \alpha_i |x_1 - x_2|,$$

$$|g_i(x_1) - g_i(x_2)| \leq \beta_i |x_1 - x_2|,$$

for all $x_1, x_2 \in R$, and $i = 1, 2, \ldots, n$.

It can be easily seen that the recurrent neural networks described by system (2) are an extension of system (1), and include CNNs, DCNNs, Hopfield networks and BAM networks. To best our knowledge, few authors have considered the global exponential stability and periodic solutions of the recurrent neural networks (2). The purpose of this Letter is to study the global exponential stability and periodic solutions of the recurrent neural networks (2), by constructing new Lyapunov functions, applying topological degree theory and utilizing nonsingular M-matrix, and our theorems do not require the activation functions to be differentiable, bounded and monotone nondecreasing. In addition these criteria are easily checked in practice, they possess important leading significance in the design and application of the recurrent neural networks. Our results extended or improved the results in [3–25].

2. Global exponential stability of delayed recurrent neural networks

Consider the special case of (2) as $I_i(t) = I_i$, i.e.,

$$x_i'(t) = -c_i h_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij})) + I_i, \quad i = 1, 2, \ldots, n.$$

(3)

Now we introduce these notations:

$$C = \operatorname{diag}(c_1, c_2, \ldots, c_n), \quad \gamma = \operatorname{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n), \quad A = (a_{ij})_{n \times n}, \quad A^+ = (a_{ij})_{n \times n},$$

$$B = (b_{ij})_{n \times n}, \quad B^+ = (b_{ij})_{n \times n}, \quad \alpha = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad \beta = \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_n).$$
Definition 1. A matrix \( A = (a_{ij})_{n \times n} \) is said to be a nonsingular M-matrix, if \( A \) has the form 
\[
sI - B, \quad s > 0, \quad B \succ 0,
\]
for which \( s > \rho(B) \), the spectral radius of \( B \).

Let \( Z^{n \times n} = \{ A = (a_{ij}) \in \mathbb{R}^{n \times n} \mid a_{ii} > 0, a_{ij} \leq 0, i \neq j \} \), from [13] we have: if \( A \in Z^{n \times n} \) is a nonsingular M-matrix, then it is equivalent to one of the following conditions:

(I) All of the principal minors of \( A \) is positive;

(II) The real part of each eigenvalue of \( A \) is positive;

(III) There exists \( r_j > 0 \), such that
\[
\frac{1}{n} \sum_{j=1}^{n} a_{ij} r_j > 0, \quad i = 1, 2, \ldots, n;
\]

(IV) There exists \( r_j > 0 \), such that
\[
\frac{1}{n} \sum_{j=1}^{n} a_{ij} r_j > 0, \quad i = 1, 2, \ldots, n.
\]

From the above equivalent conditions, we can easily check
\[
\begin{pmatrix}
4 & -1 & -2 & -3 \\
-1 & 3 & -2 & -1 \\
-0.5 & -1 & 4 & 0 \\
0 & -1 & -1 & 5
\end{pmatrix}
\]
is a nonsingular M-matrix. For more detail information one can also see [34].

Definition 2. If \( f(t) : \mathbb{R} \to \mathbb{R} \) is a continuous function, then \( D^+ f(t) \) is defined as
\[
\frac{D^+ f(t)}{dt} = \lim_{h \to 0^+} \frac{1}{h} (f(t+h) - f(t)).
\]

Definition 3. Assuming that \( f(x) : \Omega \to \mathbb{R}^n \) is a continuous and differentiable function, if \( p \notin f(\partial \Omega) \) and \( J_f(x) \neq 0, \forall x \in f^{-1}(p) \), then
\[
\deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} \text{sgn} J_f(x),
\]
where \( \Omega \subset \mathbb{R}^n \) is bounded and open, \( J_f(x) = \det(f_i, \frac{\partial f_i}{\partial x_j}) \).

Suppose \( f(x) : \Omega \to \mathbb{R}^n \) is a continuous function and \( g(x) : \Omega \to \mathbb{R}^n \) is a continuous and differentiable function, if \( p \notin f(\partial \Omega) \) and \( \|f(x) - g(x)\| < \rho(p, f(\partial \Omega)) \), then
\[
\deg(f, \Omega, p) = \deg(g, \Omega, p).
\]
For example, \( \deg(i_d, \Omega, p) = 1 \), if \( p \in \Omega \), where \( i_d(x) = x, \forall x \in \Omega \).
Definition 4. The equilibrium \(x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T\) of system (2) is said to be globally exponentially stable, if there exists constants \(c > 0\) and \(K \geq 1\) such that
\[
\sum_{i=1}^{n} |x_i(t) - x_i^*| \leq K \|\phi - x^*\| e^{-ct},
\]
for all \(t \geq 0\), where \(x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T\) is a solution with initial value \(x_i(0) = \phi_i(0), \ \forall t \in [-\tau, 0]\), \(i = 1, 2, \ldots, n\). If \(\psi(t): [-\tau, 0] \to \mathbb{R}^n\) is continuous function by \(\psi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))^T\), and its norm defined by \(\|\psi\| = \sup_{-\tau \leq t \leq 0} \sum_{i=1}^{n} |\phi_i(t)|\).

For some related definitions and stability theory, see also [26,27]. We can prove the following main results.

Theorem 1. For the system (3), suppose that \(h_1, f_i, g_i\) satisfy the hypotheses (H1) and (H2) above. Assume furthermore that the system (3) satisfies

(H3): \(C\gamma - (A^+)\alpha - (B^+)\beta\) is a nonsingular M-matrix.

Then system (3) has a unique equilibrium point \(x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T\) and it is globally exponentially stable.

Proof. If \(x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T\) denotes an equilibrium of system (3), then \(x^*\) satisfies
\[
- c_i h_i(x_i^*) + \sum_{j=1}^{n} a_{ij} f_j(x_j^*) + \sum_{j=1}^{n} b_{ij} g_j(x_j^*) + I_i = 0.
\] (4)
Rewrite (4) as
\[
CH(x^*) - AF(x^*) - BG(x^*) - I = 0,
\]
where
\[
H(x^*) = (h_1(x_1^*), h_2(x_2^*), \ldots, h_n(x_n^*))^T, \quad F(x^*) = (f_1(x_1^*), f_2(x_2^*), \ldots, f_n(x_n^*))^T, \\
G(x^*) = (g_1(x_1^*), g_2(x_2^*), \ldots, g_n(x_n^*))^T, \quad I = (I_1, I_2, \ldots, I_n)^T.
\]
Let
\[
f(x) = CH(x) - AF(x) - BG(x) - I = 0,
\] (5)
obviously, the solutions of (5) are the equilibrium of system (3). Let us define homotopic mapping
\[
F(x, \lambda) = \lambda f(x) + (1 - \lambda) h(x),
\] (6)
where \(\lambda \in [0, 1], h(x) = (h_1(x_1), h_2(x_2), \ldots, h_n(x_n))^T\). If \(F(x, \lambda) = (F_1(x, \lambda), F_2(x, \lambda), \ldots, F_n(x, \lambda))^T\), then it follows from conditions (H1), (H2) that for \(1 \leq i \leq n\)
\[
|F_i(x, \lambda)| = \left| \lambda \left[ c_i h_i(x_i) + \sum_{j=1}^{n} a_{ij} f_j(x_j) - \sum_{j=1}^{n} b_{ij} g_j(x_j) - I_i \right] + (1 - \lambda) h_i(x_i) \right|
\geq \left| \lambda c_i h_i(x_i) + (1 - \lambda) h_i(x_i) \right| - \lambda \sum_{j=1}^{n} |a_{ij}| |f_j(x_j)| - \lambda \sum_{j=1}^{n} |b_{ij}| |g_j(x_j)| - \lambda |I_i| \]
\[ \geq \left[ 1 + \lambda (c_i - 1) \right] |h_i(x_i)| - \lambda \sum_{j=1}^{n} |a_{ij}| |x_j| - \lambda \sum_{j=1}^{n} \beta_j |b_{ij}| |x_j| \]

\[ - \lambda \left[ |I_i| + \sum_{j=1}^{n} |a_{ij}| |f_j(0)| + \sum_{j=1}^{n} |b_{ij}| |g_j(0)| \right]. \]

Since \( C\gamma - (A^+)\alpha - (B^+)\beta \) is a nonsingular M-matrix, hence there exist constants \( r_i > 0 \), \( (i = 1, 2, \ldots, n) \) such that

\[ r_i c_i \gamma_i - \sum_{j=1}^{n} r_j \alpha_j |a_{ij}| - \sum_{j=1}^{n} r_j \beta_j |b_{ij}| > 0, \quad \text{or} \quad r_i c_i \gamma_i - \sum_{j=1}^{n} r_j \alpha_j |a_{ij}| - \sum_{j=1}^{n} r_j \beta_j |b_{ij}| > 0, \quad (7) \]

so

\[ \sum_{i=1}^{n} r_i |F_i(x, \lambda)| \geq \sum_{i=1}^{n} r_i (1 - \lambda) |h_i(x_i)| + \lambda \sum_{i=1}^{n} r_i c_i |h_i(x_i)| - r_i \sum_{j=1}^{n} \alpha_j |a_{ij}| |x_j| - r_i \sum_{j=1}^{n} \beta_j |b_{ij}| |x_j| \]

\[ - \lambda \sum_{i=1}^{n} r_i \left[ |I_i| + \sum_{j=1}^{n} |a_{ij}| |f_j(0)| + \sum_{j=1}^{n} |b_{ij}| |g_j(0)| \right] \]

\[ \geq \lambda \sum_{i=1}^{n} \left[ r_i c_i \gamma_i |x_i| - r_i \sum_{j=1}^{n} \alpha_j |a_{ij}| |x_j| - r_i \sum_{j=1}^{n} \beta_j |b_{ij}| |x_j| \right] \]

\[ - \lambda \sum_{i=1}^{n} \left[ |I_i| + \sum_{j=1}^{n} |a_{ij}| |f_j(0)| + \sum_{j=1}^{n} |b_{ij}| |g_j(0)| \right] \]

\[ = \lambda \sum_{i=1}^{n} \left[ r_i c_i \gamma_i - \sum_{j=1}^{n} r_j |a_{ji}| |\alpha_i| - \sum_{j=1}^{n} r_j |b_{ji}| |\beta_i| \right] |x_i| \]

\[ - \lambda \sum_{i=1}^{n} \left[ |I_i| + \sum_{j=1}^{n} |a_{ij}| |f_j(0)| + \sum_{j=1}^{n} |b_{ij}| |g_j(0)| \right]. \]

Define

\[ r_0 = \min_{1 \leq i \leq n} \left\{ r_i c_i \gamma_i - \sum_{j=1}^{n} r_j |a_{ji}| |\alpha_i| - \sum_{j=1}^{n} r_j |b_{ji}| |\beta_i| \right\}, \]

\[ I_0 = \max_{1 \leq i \leq n} \left\{ r_i \left( |I_i| + \sum_{j=1}^{n} |a_{ij}| |f_j(0)| + \sum_{j=1}^{n} |b_{ij}| |g_j(0)| \right) \right\}, \]

and let

\[ U(R_0) = \left\{ x : |x_i| < R_0 = \frac{n(I_0 + 1)}{r_0}, \quad i = 1, 2, \ldots, n \right\}, \quad (8) \]

then it follows from (8) that for any \( x \in \partial U(R_0) \) there exists \( 1 \leq i_0 \leq n \), such that

\[ |x_{i_0}| = \frac{n(I_0 + 1)}{r_0}, \]
so \[
\sum_{i=1}^{n} r_i |F_i(x, \lambda)| \geq \lambda \left( r_{i_0} c_{i_0} y_{i_0} - \sum_{j=1}^{n} r_j |a_{j,i_0}| x_{i_0} - \sum_{j=1}^{n} r_j |b_{j,i_0}| \beta_{i_0} \right) |x_{i_0}| - \lambda \sum_{i=1}^{n} I_0 \\
\geq \lambda r_{i_0} |x_{i_0}| - \lambda n I_0 = \lambda r_{i_0} \frac{n(I_0 + 1)}{r_0} - \lambda n I_0 > 0, \quad \forall \lambda \in (0, 1],
\]
that is \( F(x, \lambda) \neq 0 \), for any \( x \in \partial U(R_0) \), \( \lambda \in (0, 1] \).

If \( \lambda = 0 \), we have \( F(x, \lambda) = h(x) \neq 0 \), for any \( x \in \partial U(R_0) \). Hence \( F(x, \lambda) \neq 0 \), for any \( x \in \partial U(R_0), \lambda \in [0, 1] \).

From (H1), it is easy to prove \( \text{deg}(h, U(R_0), 0) = 1 \) where \( \text{deg}(h, U(R_0), 0) \) is topological degree. From homotopy invariance theorem \([28]\), we have
\[
\text{deg}(f, U(R_0), 0) = \text{deg}(h, U(R_0), 0) = 1.
\]
By topological degree theory, we can conclude that (5) has at least a solution in \( U(R_0) \). That is, system (3) has at least an equilibrium.

Suppose \( y^* = (y_1^*, y_2^*, \ldots, y_n^*)^T \) is also an equilibrium of system (3), then we have
\[
-c_i h_i(x_i^*) + \sum_{j=1}^{n} a_{i,j} f_j(x_j^*) + \sum_{j=1}^{n} b_{i,j} g_j(x_j^*) + l_i = 0,
\]
\[
-c_i h_i(y_i^*) + \sum_{j=1}^{n} a_{i,j} f_j(y_j^*) + \sum_{j=1}^{n} b_{i,j} g_j(y_j^*) + l_i = 0,
\]
this implies
\[
c_i (h_i(x_i^*) - h_i(y_i^*)) = \sum_{j=1}^{n} a_{i,j} (f_j(x_j^*) - f_j(y_j^*)) + \sum_{j=1}^{n} b_{i,j} (g_j(x_j^*) - g_j(y_j^*)), \quad i = 1, 2, \ldots, n,
\]
and according to (H1), (H2)
\[
c_i |x_i^* - y_i^*| \leq \sum_{j=1}^{n} a_{i,j} |x_j^* - y_j^*| + \sum_{j=1}^{n} b_{i,j} |y_j^* - y_j^*|, \quad i = 1, 2, \ldots, n. \tag{9}
\]
Rewrite (9) as
\[
(C\gamma - (A^+ \alpha - (B^+ \beta)(|x_1^* - y_1^*, |x_2^* - y_2^*, \ldots, |x_n^* - y_n^*|)^T \leq 0.
\]
It is easy to see from (H3) that \( x^* = y^* \). Hence system (3) has a unique equilibrium.

In the following, we will prove that the equilibrium point is globally exponentially stable. Make a transformation for system (3): \( z_i(t) = x_i(t) - x_i^*, \quad i = 1, 2, \ldots, n \), we have
\[
z_i'(t) = -c_i [h_i(z_i(t) + x_i^*) - h_i(x_i^*)] + \sum_{j=1}^{n} a_{i,j} [f_j(z_j(t) + x_j^*) - f_j(x_j^*)]
\]
\[
+ \sum_{j=1}^{n} b_{i,j} [g_j(z_j(t - \tau_{ij}) + x_j^*) - g_j(x_j^*)], \quad i = 1, 2, \ldots, n. \tag{10}
\]
First, we will prove the boundedness of the solutions of system (10). Let \( Z(t) = (z_1(t), z_2(t), \ldots, z_n(t))^T \) be a solution of system (10) with initial value
\[
z_i(t) = \Phi_i(t), \quad -\tau \leq t \leq 0, \quad i = 1, 2, \ldots, n,
where $\Phi(t) = (\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t))^T \in C([-\tau, 0], \mathbb{R}^n)$, $\tau = \max_{1 \leq i, j \leq n} \{\tau_{ij}\}$.

Since
$$\lim_{t \to +\infty} e^{-c_i\gamma_i t} |z_i(0)| = 0, \quad i = 1, 2, \ldots, n,$$
there exists constant $T > 0$ such that
$$\frac{1}{r_i} e^{-c_i\gamma_i t} |z_i(0)| < 1 - \delta_i, \quad i = 1, 2, \ldots, n, \forall t \geq T,$$
where
$$\delta_i = \frac{\sum_{j=1}^n r_j \alpha_j |a_{ij}| + \sum_{j=1}^n r_j \beta_j |b_{ij}|}{r_i c_i \gamma_i} < 1.$$

By continuity of $z_i(t)$ and boundedness of $z_i(t)$ on $[-\tau, 0]$, there exists
$$N \geq \max_{1 \leq i \leq n} \left\{1, \frac{1}{r_i} \sup_{-\tau \leq t \leq 0} (|z_i(t)| + 1)\right\},$$
such that
$$|z_i(t)| < r_i N, \quad i = 1, 2, \ldots, n, \forall t \in [-\tau, T].$$

Then we will show that
$$|z_i(t)| < r_i N \leq \max_{1 \leq i \leq n} \{r_i N\}, \quad i = 1, 2, \ldots, n, \forall t \in [-\tau, +\infty). \quad (11)$$

If (11) is not true, there must be some $i \in \{1, 2, \ldots, n\}$ and $t^* > T$, such that
$$\frac{1}{r_i} |z_i(t^*)| = N, \quad \frac{1}{r_j} |z_j(t)| \leq N, \quad j = 1, 2, \ldots, n, \forall t \leq t^*. \quad (12)$$

From (10), we have
$$\frac{D^+ |z_i(t)|}{dt} = \text{sgn} z_i(t) \frac{dz_i(t)}{dt}$$
$$= \text{sgn} z_i(t) \left[ -c_i \left(h_i(z_i(t) + x_i^*) - h_i(x_i^*)\right) + \sum_{j=1}^n a_{ij} \left(f_j(z_j(t) + x_j^*) - f_j(x_j^*)\right) \right.$$
$$\left. + \sum_{j=1}^n b_{ij} \left(g_j(z_j(t - \tau_{ij}) + x_j^*) - g_j(x_j^*)\right)\right]$$
$$\leq -c_i \gamma_i |z_i(t)| + \sum_{j=1}^n |a_{ij}| |f_j(z_j(t) + x_j^*) - f_j(x_j^*)| + \sum_{j=1}^n |b_{ij}| |g_j(z_j(t - \tau_{ij}) + x_j^*) - g_j(x_j^*)|$$
$$\leq -c_i \gamma_i |z_i(t)| + \sum_{j=1}^n \alpha_j |a_{ij}| |z_j(t)| + \sum_{j=1}^n \beta_j |b_{ij}| |z_j(t - \tau_{ij})|,$$
so
$$\frac{D^+ |z_i(t)|}{dt} \leq -c_i \gamma_i |z_i(t)| + \sum_{j=1}^n \alpha_j |a_{ij}| |z_j(t)| + \sum_{j=1}^n \beta_j |b_{ij}| |z_j(t - \tau_{ij})|$$
$$\leq -c_i \gamma_i |z_i(t)| + \sum_{j=1}^n \alpha_j |a_{ij}| r_j N + \sum_{j=1}^n \beta_j |b_{ij}| r_j N, \quad \forall t \leq t^*, \quad (13)$$
Now we consider the Lyapunov function

\[ V(t) = \sum_{i=1}^{n} r_i \left[ |z_i(t)| e^{ct} + \sum_{j=1}^{n} |b_{ij}| \beta_j \left( \int_{1-t_{ij}}^{t} |z_j(s)| e^{c(t+s)} \, ds \right) \right]. \] (13)

Calculating the upper right derivative of \( V \) along (10), we have

\[
\frac{D^+ V(t)}{dt} \leq \sum_{i=1}^{n} r_i \left( \text{sgn} \, z_i(t) \frac{dz_i(t)}{dt} e^{ct} + |z_i(t)| c e^{ct} + \sum_{j=1}^{n} |b_{ij}| \beta_j \left( |z_j(t)| e^{c(t+\tau_{ij})} - |z_j(t-\tau_{ij})| e^{ct} \right) \right)
\]

\[
\leq \sum_{i=1}^{n} r_i \left( -c_i \gamma_i |z_i(t)| e^{ct} + \sum_{j=1}^{n} |a_{ij}| |z_j(t)| e^{ct} + \sum_{j=1}^{n} |b_{ij}| |z_j(t-\tau_{ij})| e^{ct} \right) + |z_i(t)| c e^{ct} + \sum_{j=1}^{n} |b_{ij}| \beta_j \left( |z_j(t)| e^{c(t+\tau_{ij})} - |z_j(t-\tau_{ij})| e^{ct} \right)
\]

\[
= \sum_{i=1}^{n} r_i \left( -c_i \gamma_i |z_i(t)| e^{ct} + |z_i(t)| c e^{ct} + \sum_{j=1}^{n} |a_{ij}| |z_j(t)| e^{ct} + \sum_{j=1}^{n} |b_{ij}| |z_j(t)| e^{c(t+\tau_{ij})} \right)
\]

\[
eq e^{ct} \sum_{i=1}^{n} r_i \left[ -c_i \gamma_i + c \right] |z_i(t)| + \sum_{j=1}^{n} |a_{ij}| |z_j(t)| + e^{ct} \sum_{j=1}^{n} |b_{ij}| |z_j(t)|
\]

\[
eq e^{ct} \sum_{j=1}^{n} \left( -r_j(c_j \gamma_j - c) + \sum_{i=1}^{n} r_i |a_{ij}| + e^{ct} \sum_{i=1}^{n} r_i |b_{ij}| \right) |z_j(t)|
\]

\[
\leq 0,
\]

so

\[ V(t) \leq V(0). \]
Since
\[ e^{ct} \min_{1 \leq i \leq n} \{ r_i \} \sum_{i=1}^{n} |z_i(t)| \leq V(t), \quad \forall t \geq 0, \]
\[ V(0) = \sum_{i=1}^{n} r_i \left( |\Phi_i(0)| + \sum_{j=1}^{n} |b_{ij}| \beta_j \int_{-\tau_{ij}}^{0} |\Phi_j(s)| e^{c(t+s)} \, ds \right) \]
\[ \leq \max_{1 \leq i \leq n} \{ r_i \} \| \Phi \| + \sum_{i=1}^{n} r_i \sum_{j=1}^{n} |b_{ij}| \beta_j e^{cT} \| \Phi \| \]
\[ = \left( \max_{1 \leq i \leq n} \{ r_i \} + \sum_{i=1}^{n} r_i \sum_{j=1}^{n} |b_{ij}| \beta_j \right) \| \Phi \|. \]

Then we easily get
\[ \sum_{i=1}^{n} |z_i(t)| \leq \frac{\max_{1 \leq i \leq n} \{ r_i \} + \sum_{i=1}^{n} r_i \sum_{j=1}^{n} |b_{ij}| \beta_j \tau e^{cT}}{\min_{1 \leq i \leq n} \{ r_i \}} \| \Phi \| e^{-ct}, \]
for all \( t \geq 0 \). This implies that the equilibrium of system (10) is globally exponentially stable, i.e., the equilibrium of system (3) is globally exponentially stable. This completes the proof. \( \square \)

3. Periodic solutions of delayed recurrent neural networks

In this section, we study the periodic solutions of the following system
\[ x_i'(t) = -c_i h_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij})) + I_i(t), \quad \text{(14)} \]
where \( c_i > 0, \quad I_i(t) : R^+ \rightarrow R \) are continuously periodic functions with period \( \omega, \quad i = 1, 2, \ldots, n \), other symbols possess the same meaning as that of (3).

**Theorem 2.** There exists exactly one \( \omega \)-periodic solution of system (14) and all other solutions of system (14) converge exponentially to it as \( t \rightarrow + \infty \), if (H1)–(H3) hold.

**Proof.** For any \( \Phi \in C([-\tau, 0], R^n) \), we define
\[ \| \Phi \| = \sup_{-\tau \leq t \leq 0} \sum_{i=1}^{n} |\Phi_i(t)|. \]

For any \( \varphi, \psi \in C([-\tau, 0], R^n) \), we denote the solutions of system (14) through \( (0, \varphi) \) and \( (0, \psi) \) as \( x(t, \varphi) = (x_1(t, \varphi), x_2(t, \varphi), \ldots, x_n(t, \varphi))^T, \quad x(t, \psi) = (x_1(t, \psi), x_2(t, \psi), \ldots, x_n(t, \psi))^T \), respectively. Define
\[ x_t(\varphi) = x(t + \theta, \varphi), \quad \theta \in [-\tau, 0], \quad t \geq 0, \]
then \( x_t(\varphi) \in C([-\tau, 0], R^n) \) for all \( t \geq 0 \).
Thus it follows from system (14) that

\[
\left( x_i(t, \varphi) - x_i(t, \psi) \right)' = -c_i \left( h_i(x_i(t, \varphi)) - h_i(x_i(t, \psi)) \right) + \sum_{j=1}^{n} a_{ij} \left[ f_j(x_j(t, \varphi)) - f_j(x_j(t, \psi)) \right] + \sum_{j=1}^{n} b_{ij} \left[ g_j(x_j(t - \tau_j, \varphi)) - g_j(x_j(t - \tau_j, \psi)) \right],
\]

for all \( t \geq 0, i = 1, 2, \ldots, n \).

Now we consider another Lyapunov function

\[
V(t) = \sum_{i=1}^{n} r_i \left( |x_i(t, \varphi) - x_i(t, \psi)| e^{ct} + \sum_{j=1}^{n} |b_{ij}| \beta_j \int_{t-\tau_i}^{t} |x_j(s, \varphi) - x_j(s, \psi)| e^{c(s - \tau_i)} \, ds \right).
\]

By a minor modification of the proof of Theorem 1, we can easily get

\[
\sum_{i=1}^{n} |x_i(t, \varphi) - x_i(t, \psi)| \leq \max_{1 \leq i \leq n} \{ r_i \} + \sum_{i=1}^{n} r_i \sum_{j=1}^{n} |b_{ij}| \beta_j \tau_i e^{ct} \| \varphi - \psi \| e^{-ct} = K e^{-ct} \| \varphi - \psi \|,
\]

for all \( t \geq 0 \), where

\[
K = \frac{\max_{1 \leq i \leq n} \{ r_i \} + \sum_{i=1}^{n} r_i \sum_{j=1}^{n} |b_{ij}| \beta_j \tau_i e^{ct}}{\min_{1 \leq i \leq n} \{ r_i \}} \geq 1,
\]

one can easily obtain from the formula above that

\[
\| x_t(\varphi) - x_t(\psi) \| \leq K e^{-c(t - \tau)} \| \varphi - \psi \|,
\]

we can choose a positive integer \( m \) such that

\[
K e^{-c(m\omega - \tau)} \leq \frac{1}{2}.
\]

Now define a Poincaré mapping \( P : C([-\tau, 0], R^n) \to C([-\tau, 0], R^n) \) by \( P\varphi = x_\omega(\varphi) \), then we have

\[
\| P^m \varphi - P^m \psi \| \leq \frac{1}{2} \| \varphi - \psi \|.
\]

This implies that \( P^m \) is a contraction mapping, hence there exists a unique fixed point \( \phi^* \in C([-\tau, 0], R^n) \) such that \( P^m \phi^* = \phi^* \).

Note that

\[
P^m (P\varphi) = P(P^m \varphi) = P\varphi,
\]

this shows that \( P\phi^* \in C([-\tau, 0], R^n) \) is also a fixed point of \( P^m \), so

\[
P\phi^* = \phi^*, \quad \text{i.e.,} \quad x_\omega(\phi^*) = \phi^*.
\]

Let \( x(t, \phi^*) \) be the solution of system (14) through \((0, \phi^*)\), obviously \( x(t + \omega, \phi^*) \) is also a solution of system (14), and note that

\[
x_{t + \omega}(\phi^*) = x_t(x_{\omega}(\phi^*)) = x_t(\phi^*),
\]

for all \( t \geq 0 \).
Therefore
\[ x(t + \omega, \phi^*) = x(t, \phi^*), \quad \forall t \geq 0. \]

This shows that \( x(t, \phi^*) \) is exactly one \( \omega \)-periodic solution of system (14) and from (15) it is easy to see that all solutions of system (14) converge exponentially to it as \( t \to +\infty \). This completes the proof. \( \square \)

**Remark 1.** We can easily see that system (1) is a special case of system (2) when \( h_l(x_l(t)) = x_l(t), \ g_j(x_j(t - \tau_j)) = f_j(x_j(t - \tau_j)) \) in system (2), thus the results of this Letter can be applied to the models of CNNs and DCNNs. In addition, our results need only the activation functions satisfy \((H_2)\), not requiring the activation functions to be differentiable, bounded and monotone nondecreasing.

4. Two illustrative examples

**Example 1.** Let \( n = 2, h_1(x) \equiv h(x) = x + e^x - 1, \ f_j(x) \equiv f(x) = \arctan x, \ g_j(x) \equiv g(x) = \frac{1}{2}(|x + 1| - |x - 1|), \) obviously, \( \gamma_l = \inf_{x \in R} h_l'(x) = 1, \ h_l(0) = 0, \ i = 1, 2, \ \alpha_j = \beta_j = 1, \ j = 1, 2. \) Consider the neural networks with delays
\[
\begin{align*}
x_1'(t) &= -c_1(x_1 + e^{x_1} - 1) + a_{11} f(x_1(t)) + a_{12} f(x_2(t)) + b_{11} g(x_1(t - \tau_{11})) + b_{12} g(x_2(t - \tau_{12})) + I_1, \\
x_2'(t) &= -c_2(x_2 + e^{x_2} - 1) + a_{21} f(x_1(t)) + a_{22} f(x_2(t)) + b_{21} g(x_1(t - \tau_{21})) + b_{22} g(x_2(t - \tau_{22})) + I_2.
\end{align*}
\]

By taking \( c_1 = 9, c_2 = 8, a_{11} = a_{22} = 1, a_{12} = 2, a_{21} = 3, b_{11} = 3, b_{12} = 1, b_{21} = 2, b_{22} = -2 \), we obtain
\[
C \gamma - (A^+) \alpha - (B^+) \beta = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}
\]
is a nonsingular M-matrix. Hence the equilibrium is globally exponentially stable.

**Example 2.** Let \( n = 2, h_1(x) \equiv h(x) = x + e^x - 1, \ f_j(x) \equiv f(x) = \arctan x, \ g_j(x) \equiv g(x) = \frac{1}{2}(|x + 1| - |x - 1|), \) obviously, \( \gamma_l = \inf_{x \in R} h_l'(x) = 1, \ h_l(0) = 0, \ i = 1, 2, \ \alpha_j = \beta_j = 1, \ j = 1, 2. \) Consider the neural networks with delays
\[
\begin{align*}
x_1'(t) &= -c_1(x_1 + e^{x_1} - 1) + a_{11} f(x_1(t)) + a_{12} f(x_2(t)) + b_{11} g(x_1(t - \tau_{11})) + b_{12} g(x_2(t - \tau_{12})) + \sin t, \\
x_2'(t) &= -c_2(x_2 + e^{x_2} - 1) + a_{21} f(x_1(t)) + a_{22} f(x_2(t)) + b_{21} g(x_1(t - \tau_{21})) + b_{22} g(x_2(t - \tau_{22})) + \cos t.
\end{align*}
\]

By taking \( c_1 = 9, c_2 = 10, a_{11} = a_{22} = 2, a_{12} = 1, a_{21} = -3, b_{11} = 1, b_{12} = -2, b_{21} = 2, b_{22} = 4 \), we have
\[
C \gamma - (A^+) \alpha - (B^+) \beta = \begin{pmatrix} 6 \\ -5 \\ 4 \end{pmatrix}
\]
is a nonsingular M-matrix. From Theorem 2, the equation has one \( 2\pi \)-periodic solution and all other solutions converge exponentially to it as \( t \to +\infty \).

5. Conclusion

A set of criteria have been derived ensuring the global exponential stability (GES) and periodic solutions of a class of delayed recurrent neural networks with more general activation functions by introducing nonsingular M-matrix, constructing suitable Lyapunov functional and applying topological degree theory. These criteria are independent of delays. The method of this Letter is also suitable to delayed Hopfield neural networks, CNNs, DCNNs and BAM networks.
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