Multiperiodicity and Exponential Attractivity Evoked by Periodic External Inputs in Delayed Cellular Neural Networks

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We show that an $n$-neuron cellular neural network with time-varying delay can have $2^n$ periodic orbits located in saturation regions and these periodic orbits are locally exponentially attractive. In addition, we give some conditions for ascertaining periodic orbits to be locally or globally exponentially attractive and allow them to locate in any designated region. As a special case of exponential periodicity, exponential stability of delayed cellular neural networks is also characterized. These conditions improve and extend the existing results in the literature. To illustrate and compare the results, simulation results are discussed in three numerical examples.

1 Introduction

Cellular neural networks (CNNs) and delayed cellular neural networks (DCNNs) are arrays of dynamical cells that are suitable for solving many complex computational problems. In recent years, both have been extensively studied and successfully applied for signal processing and solving nonlinear algebraic equations.

As dynamic systems with a special structure, CNNs and DCNNs have many interesting properties that deserve theoretical studies. In general, there are two interesting nonlinear neurodynamic properties in CNNs and DCNNs: stability and periodic oscillations. The stability of a CNN or a DCNN at an equilibrium point means that for a given activation function and a constant input vector, an equilibrium of the network exists and any state in the neighborhood converges to the equilibrium. The stability of neuron activation states at an equilibrium is prerequisite for most applications. Some neurodynamics have multiple (two) stable equilibria and may be stable at any equilibrium depending on the initial state, which is called
multistability (bistability). For stability, either an equilibrium or a set of equilibria is the attractor.

Besides stability, an activation state may be periodically oscillatory around an orbit. In this case, the attractor is a limit set. Periodic oscillation in recurrent neural networks is an interesting dynamic behavior, as many biological and cognitive activities (e.g., heartbeat, respiration, mastication, locomotion, and memorization) require repetition. Persistent oscillation, such as limit cycles, represents a common feature of neural firing patterns produced by dynamic interplay between cellular and synaptic mechanisms. Stimulus-evoked oscillatory synchronization was observed in many biological neural systems, including the cerebral cortex of mammals and the brain of insects. It was also known that time delays can cause oscillations in neurodynamics (Gopalsamy & Leung, 1996; Belair, Campbell, & Driessche, 1996). In addition, periodic oscillations in recurrent neural networks have found many applications, such as associative memories (Nishikawa, Lai, & Hoppensteadt, 2004), pattern recognition (Wang, 1995; Chen, Wang, & Liu, 2000), machine learning (Ruiz, Owens, & Townley, 1998; Townley et al., 2000), and robot motion control (Jin & Zacksenhouse, 2003). The analysis of periodic oscillation of neural networks is more general than stability analysis since an equilibrium point can be viewed as a special case of oscillation with any arbitrary period.

The stability of CNNs and DCNNs has been widely investigated (e.g., Chua & Roska, 1990, 1992; Civalleri, Gilli, & Pandolfi, 1993; Liao, Wu, & Yu, 1999; Roska, Wu, Balsi, & Chua, 1992; Roska, Wu, & Chua, 1993; Setti, Thiran, & Serpico, 1998; Takahashi, 2000; Zeng, Wang, & Liao, 2003). The existence of periodic orbits together with global exponential stability of CNNs and DCNNs is studied in Yi, Heng, and Vadakkepat (2002) and Wang and Zou (2004). Most existing studies (Berns, Moiola, & Chen, 1998; Jiang & Teng, 2004; Kanamaru & Sekeine, 2004; Liao & Wang, 2003; Liu, Chen, Cao, & Huang, 2003; Wang & Zou, 2004) are based on the assumption that the equilibrium point of CNNs or DCNNs is globally stable or the periodic orbit of CNNs or DCNNs is globally attractive; hence, CNNs or DCNNs have only one equilibrium point or one periodic orbit. However, in most applications, it is required that CNNs or DCNNs exhibit more than one stable equilibrium point (e.g., Yi, Tan, & Lee, 2003; Zeng, Wang, & Liao, 2004), or more than one exponentially attractive periodic orbit instead of a single globally stable equilibrium point.

In this letter, we investigate the multiperiodicity and multistability of DCNNs. We show that an \( n \)-neuron DCNN can have \( 2^n \) periodic orbits that are locally exponentially attractive. Moreover, we present the estimates of attractive domain of such \( 2^n \) locally exponentially attractive periodic orbits. In addition, we give the conditions for periodic orbits to be locally or globally exponentially attractive when the periodic orbits locate in a designated position. All of these conditions are very easy to be verified.
The remaining part of this letter consists of six sections. In section 2, relevant background information is given. The main results are stated in sections 3, 4, and 5. In section 6, three illustrative examples are provided with simulation results. Finally, concluding remarks are given in section 7.

## 2 Preliminaries

Consider the DCNN model governed by the following normalized dynamic equations:

\[
\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{n} a_{ij}f(x_j(t)) + \sum_{j=1}^{n} b_{ij}f(x_j(t - \tau_j(t))) + u_i(t), i = 1, \ldots, n, \tag{2.1}
\]

where \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) is the state vector, \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are connection weight matrices that are not assumed to be symmetric, \( u(t) = (u_1(t), \ldots, u_n(t))^T \in \mathbb{R}^n \) is a periodic input vector with period \( \omega \) (i.e., there exists a constant \( \omega > 0 \) such that \( u_i(t + \omega) = u_i(t) \) \( \forall t \geq 0, \forall i \in \{1, 2, \ldots, n\} \)), \( \tau_j(t) \) is the time-varying delay that satisfies \( 0 \leq \tau_j(t) \leq \tau \) (\( \tau \) is constant), and \( f(\cdot) \) is the piecewise linear activation function defined by \( f(v) = |v + 1| - |v - 1|)/2 \). In particular, when \( b_{ij} \equiv 0 \) (\( \forall i, j = 1, 2, \ldots, n \)), the DCNN degenerates as a CNN.

Let \( C([t_0 - \tau, t_0], D) \) be the space of continuous functions mapping \( [t_0 - \tau, t_0] \) into \( D \subset \mathbb{R}^n \) with the norm defined by \( ||\phi||_{t_0} = \max_{1 \leq i \leq n} \{\sup_{t_0 - \tau \leq t \leq t_0} ||\phi_i(t)||\} \), where \( \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \). Denote \( ||x|| = \max_{1 \leq i \leq n} ||x_i|| \) as the vector norm of the vector \( x = (x_1, \ldots, x_n)^T \).

\( \forall \phi, \varphi \in C([t_0 - \tau, t_0], D) \), where \( \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \), \( \varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s))^T \). Denote \( ||\phi, \varphi||_{t_0} = \max_{1 \leq i \leq n} \{\sup_{t_0 - \tau \leq t \leq t_0} ||\phi_i(s) - \varphi_i(s)||\} \) as a measurement in \( C([t_0 - \tau, t_0], D) \).

The initial condition of DCNN model 2.1 is assumed to be \( \phi(\bar{t}) = (\phi_1(\bar{t}), \phi_2(\bar{t}), \ldots, \phi_n(\bar{t}))^T \), where \( \phi(\bar{t}) \in C([t_0 - \tau, \bar{t}], \mathbb{R}^n) \). Denote \( x(t; t_0, \phi) \) as the state of DCNN model 2.1 with initial condition \((t_0, \phi)\). It means that \( x(t; t_0, \phi) \) is continuous and satisfies equation 2.1 and \( x(s; t_0, \phi) = \phi(s) \), for \( s \in [t_0 - \tau, t_0] \).

Denote \( (-\infty, -1) = (-\infty, -1) \times (-1, 1]^0 \times (1, +\infty)^3; [-1, 1] = (-\infty, -1)^0 \times [-1, 1] \times (1, +\infty)^3; (1, +\infty) = (-\infty, -1) \times [-1, 1] \times (1, +\infty)^3; \mathfrak{H} = (-\infty, +\infty) \), so \((-\infty, +\infty)^n \) can be divided into \( 3^n \) subspaces:

\[
\Omega = \{\prod_{i=1}^{n} (-\infty, -1)^{\delta_{1i}} \times [-1, 1)^{\delta_{2i}} \times (1, +\infty)^{\delta_{3i}}, \delta_{1i}, \delta_{2i}, \delta_{3i} \in \{0, 1\}, i = 1, \ldots, n\}; \tag{2.2}
\]
and $\Omega$ can be divided into three subspaces:

$$
\Omega_1 = \{[-1, 1]^n\}
$$

$$
\Omega_2 = \prod_{i=1}^n (-\infty, -1)^{\delta(i)} \times (1, +\infty)^{1-\delta(i)}, \quad \delta(i) = 1 \text{ or } 0, \quad i = 1, \ldots, n
$$

$$
\Omega_3 = \Omega - \Omega_1 - \Omega_2
$$

Hence, $\Omega_1$ is composed of one region, $\Omega_2$ is composed of $2^n$ regions, and $\Omega_3$ is composed of $3^n - 2^n - 1$ regions.

**Definition 1.** A periodic orbit $x^*(t)$ is said to be a limit cycle of a DCNN if $x^*(t)$ is an isolated periodic orbit of the DCNN; that is, there exists $\omega > 0$ such that $\forall t \geq t_0, x^*(t + \omega) = x^*(t)$, and there exists $\delta > 0$ such that $\forall \tilde{x} \in [x] 0 < ||x^*(t)|| < \delta, x \in \mathbb{R}^n, t \geq t_0$, where $\tilde{x}$ is not a point on any periodic orbit of the DCNN.

**Definition 2.** A periodic orbit $x^*(t)$ of a DCNN is said to be locally exponentially attractive in region $\Xi$ if there exist constants $\alpha > 0, \beta > 0$ such that $\forall t \geq t_0$,

$$
\|x(t; t_0, \phi) - x^*(t)\| \leq \beta \|\phi\|_{t_0} \exp(-\alpha(t - t_0)),
$$

where $x(t; t_0, \phi)$ is the state of the DCNN with any initial condition $(t_0, \phi), \phi(\varnothing) \in C([t_0 - \tau, t_0], \Xi)$, and $\Xi$ is said to be a locally exponentially attractive set of the periodic orbit $x^*(t)$. When $\Xi = \mathbb{R}^n$, $x^*(t)$ is said to be globally exponentially attractive. In particular, if $x^*(t)$ is a fixed point $x^*$, then the DCNN is said to be globally exponentially stable.

**Lemma 1.** (Kosaku, 1978). Let $\mathcal{D}$ be a compact set in $\mathbb{R}^n$, $H$ be a mapping on complete metric space $(C([t_0 - \tau, t_0], \mathcal{D}), ||\cdot||_{t_0})$. If $H(C([t_0 - \tau, t_0], \mathcal{D})) \subset C([t_0 - \tau, t_0], \mathcal{D})$, and there exists a constant $\alpha < 1$ such that $\forall \phi, \varphi \in C([t_0 - \tau, t_0], \mathcal{D}), ||H(\phi), H(\varphi)||_{t_0} \leq \alpha ||\phi, \varphi||_{t_0}$, then there exists $\phi^* \in C([t_0 - \tau, t_0], \mathcal{D})$ such that $H(\phi^*) = \phi^*$.

Consider the following coupled system:

$$
\frac{dx(t)}{dt} = -x(t) + Ay(t) + By(t - \tau(t)),
$$

$$
\frac{dy(t)}{dt} = h(t, y(t), y(t - \tau(t))),
$$

where $x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^m, A$ and $B$ are $n \times m$ matrices, $h \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^m)$, and $C(\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ is the space of continuous functions mapping $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ into $\mathbb{R}^m$. 

Lemma 2. If system 2.4 is globally exponentially stable, then system 2.3 is also globally exponentially stable.

Proof. By applying the variant format of constants, the solution $x(t)$ of equation 2.3 can be expressed as

$$x(t) = \exp(-(t-t_0))x(t_0) + \int_{t_0}^{t} \exp(-(t-s))(Ay(s) + By(s - \tau(s)))ds.$$

Since equation 2.4 is globally exponentially stable, there exist constants $\alpha, \beta > 0$ such that $|y(s)| \leq \beta \exp(-\alpha(s-t_0))$. Hence, $|Ay(s) + By(s - \tau(s))| \leq \beta \exp(-\alpha(s-t_0))$, where $\beta = (||A|| + ||B|| \exp(\alpha \tau))\beta$. Then when $\alpha = 1$, $\forall t \geq t_0$, $|x(t)| \leq |x(t_0)| \exp(-(t-t_0)) + \bar{\beta}(t - t_0) \exp(-t - t_0)$; when $\alpha \neq 1$, $|x(t)| \leq |x(t_0)| \exp(-(t-t_0)) + \bar{\beta}(\exp(-(t-t_0)) + \exp(-\alpha(t-t_0)))/|1-\alpha|$; that is, equation 2.3 is also globally exponentially stable.

Throughout this article, we assume that $N_1 \cup N_2 \cup N_3 = \{1, 2, \ldots, n\}$, $N_1 \cap N_2$, $N_1 \cap N_3$, and $N_2 \cap N_3$ are empty. Denote

$${\mathcal{D}} = \{x \in \mathbb{R}^n | x_i \in (-\infty, -1), i \in N_1; x_i \in (1, \infty), i \in N_2; x_i \in [-1, 1], i \in N_3\}.$$

Note that $\mathcal{D} \subset \Omega$, where $\Omega$ is defined in equation 2.2.

If $N_3$ is empty, then denote

$${\mathcal{D}}_2 = \{x \in \mathbb{R}^n | x_i \in (-\infty, -1), i \in N_1; x_i \in (1, \infty), i \in N_2\}.$$

3 Locally Exponentially Attractive Multiperiodicity in a Saturation Region

In this section, we show that an $n$-neuron delayed cellular neural network can have $2^n$ periodic orbits located in saturation regions and these periodic orbits are locally exponentially attractive.

Theorem 1. If $\forall i \in \{1, 2, \ldots, n\}, \forall t \geq t_0$,

$$|u_i(t)| < a_{ii} - 1 - \sum_{j=1, j \neq i}^{n} |a_{ij}| - \sum_{j=i}^{n} |b_{ij}|, \quad (3.1)$$

then DCNN (see equation 2.1) has $2^n$ locally exponentially attractive limit cycles.

Proof. If $\forall s \in [t_0 - \tau, t], x(s) \in \mathcal{D}_2$, from equation 2.1, $\forall i = 1, 2, \ldots, n$,

$$\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}) + u_i(t). \quad (3.2)$$
When $i \in N_2$ and $x_i(t) = 1$, from equations 3.1 and 3.2,
\[
\frac{dx_i(t)}{dt} = -1 + \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}) + u_i(t) > 0. \tag{3.3}
\]

When $i \in N_1$ and $x_i(t) = -1$, from equations 3.1 and 3.2,
\[
\frac{dx_i(t)}{dt} = 1 + \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}) + u_i(t) < 0. \tag{3.4}
\]

Equations 3.3 and 3.4 imply that if $\forall \phi \in C([t_0 - \tau, t_0], D_2)$, then $x(t; t_0, \phi)$ will keep in $D_2$, and $D_2$ is an invariant set of DCNN (see equation 2.1). So $\forall t \geq t_0 - \tau, x(t) \in D_2$. Hence, DCNN, equation 2.1, can be rewritten as equation 3.2.

Let $x(t; t_0, \phi)$ and $x(t; t_0, \varphi)$ be two states of DCNN, equation 2.1, with initial conditions $(t_0, \phi)$ and $(t_0, \varphi)$, where $\phi, \varphi \in C([t_0 - \tau, t_0], D_2)$. From equations 2.1 and 3.2, $\forall i \in \{1, 2, \ldots, n\}$, $\forall t \geq t_0$,
\[
\frac{d(x_i(t; t_0, \phi) - x_i(t; t_0, \varphi))}{dt} = -(x_i(t; t_0, \phi) - x_i(t; t_0, \varphi)). \tag{3.5}
\]

Hence, $\forall i = 1, 2, \ldots, n$, $\forall t \geq t_0$,
\[
|x_i(t; t_0, \phi) - x_i(t; t_0, \varphi)| \leq ||\phi, \varphi||_{t_0} \exp\{-(t - t_0)\}. \tag{3.6}
\]

Define $x_{\phi}^{(t)}(\theta) = x(t + \theta; t_0, \phi)$, $\theta \in [t_0 - \tau, t_0]$. Then from equations 3.3 and 3.4, $\forall \phi \in C([t_0 - \tau, t_0], D_2)$, $x_{\phi}^{(t)} \in C([t_0 - \tau, t_0], D_2)$. Define a mapping $H : C([t_0 - \tau, t_0], D_2) \rightarrow C([t_0 - \tau, t_0], D_2)$ by $H(\phi) = x_{\phi}^{(t_0)}$. Then
\[
H(C([t_0 - \tau, t_0], D_2)) \subset C([t_0 - \tau, t_0], D_2),
\]
and $H^m(\phi) = x_{\phi}^{(m\omega)}$. We can choose a positive integer $m$ such that $\exp\{-(m\omega - \tau)\} \leq \alpha < 1$. Hence, from equation 3.6,
\[
||H^m(\phi), H^m(\varphi)||_{t_0} \leq \max_{1 \leq i \leq n} \left\{ \sup_{\theta \in [0 - \tau, t_0]} |x_i(m\omega + \theta; t_0, \phi) - x_i(m\omega + \theta; t_0, \varphi)| \right\} \\
\leq ||\phi, \varphi||_{t_0} \exp\{-(m\omega + t_0 - \tau - t_0)\} \leq \alpha ||\phi, \varphi||_{t_0}.
\]

Based on lemma 1, there exists a unique fixed point $\phi^* \in C([t_0 - \tau, t_0], D_2)$ such that $H^m(\phi^*) = \phi^*$. In addition, $H^m(H(\phi^*)) = H(H^m(\phi^*)) = H(\phi^*)$. This shows that $H(\phi^*)$ is also a fixed point of $H^m$. Hence, by the uniqueness of the fixed
point of the mapping $H$. $H(\phi^*) = \phi^*$; that is, $x_{\phi^*}(\omega) = \phi^*$. Let $x(t; t_0, \phi^*)$ be a state of DCNN, equation 2.1, with initial condition $(t_0, \phi^*)$. Then from equation 2.1, $\forall i = 1, 2, \ldots, n, \forall t \geq t_0$,

$$\frac{dx_i(t; t_0, \phi^*)}{dt} = -x_i(t; t_0, \phi^*) + \sum_{j \in N_i}(a_{ij} + b_{ij}) - \sum_{j \in N_2}(a_{ij} + b_{ij}) + u_i(t).$$

Hence, $\forall i = 1, 2, \ldots, n, \forall t + \omega \geq t_0$,

$$\frac{dx_i(t + \omega; t_0, \phi^*)}{dt} = -x_i(t + \omega; t_0, \phi^*) + \sum_{j \in N_i}(a_{ij} + b_{ij}) - \sum_{j \in N_2}(a_{ij} + b_{ij}) + u_i(t + \omega) = -x_i(t + \omega; t_0, \phi^*) + \sum_{j \in N_i}(a_{ij} + b_{ij}) - \sum_{j \in N_2}(a_{ij} + b_{ij}) + u_i(t).$$

This implies $x(t + \omega; t_0, \phi^*)$ is also a state of DCNN, equation 2.1, with initial condition $(t_0, \phi^*)$. $x_{\phi^*}(\omega) = \phi^*$ implies that $\forall t \geq t_0$,

$$x(t + \omega; t_0, \phi^*) = x(t; t_0, x_{\phi^*}(\omega)) = x(t; t_0, \phi^*).$$

Hence, $x(t; t_0, \phi^*)$ is a periodic orbit of DCNN, equation 2.1, with period $\omega$. From equation 3.5, it is easy to see that any state of DCNN, equation 2.1, with initial condition $(t, \phi)$ ($\phi \in C([t_0 - \tau, t_0], \mathcal{D}_2)$) converges to this periodic orbit exponentially as $t \rightarrow +\infty$. Hence, the isolated periodic orbit $x(t; t_0, \phi^*)$ located in $\Omega_2$ is locally exponentially attractive, and $\mathcal{D}_2$ is a locally exponentially attractive set of $x(t; t_0, \phi^*)$. Since there exist $2^n$ elements in $\Omega_2$, there exist $2^n$ isolated periodic orbits in $\Omega_2$. And such $2^n$ isolated periodic orbits are locally exponentially attractive.

When the periodic external input $u(t)$ degenerates into a constant vector, we have the following corollary:

**Corollary 1.** If $\forall i \in \{1, 2, \ldots, n\}, \forall t \geq t_0, u_i(t) \equiv u_i$ (constant), and

$$|u_i| < a_{ii} - 1 - \sum_{j=1, j \neq i}^n |a_{ij}| - \sum_{j=1}^n |b_{ij}|,$$

then DCNN (see equation 2.1) has $2^n$ locally exponentially stable equilibrium points.
Proof. Since $u_i(t) \equiv u_i$ (constant), for an arbitrary constant $\nu \in \mathbb{R}$, $u_i(t + \nu) \equiv u_i$. According to theorem 1, DCNN, equation 2.1, has $2^n$ locally exponentially attractive limit cycles with period $\nu$. The arbitrariness of constant $\nu$ implies that such limit cycles are fixed points. Hence, DCNN, equation 2.1, has $2^n$ locally exponentially attractive equilibrium points.

Remark 1. In theorem 1 and corollary 1, it is necessary for $a_{ii}$ to be dominant such that $a_{ii} > 1 + \sum_{j=1, j \neq i}^n |a_{ij}| + \sum_{j=1}^n |b_{ij}|$.

Remark 2. A main objective for designing associative memories is to store a large number of patterns as stable equilibria or limit cycles such that stored patterns can be retrieved when the initial probes contain sufficient information about the patterns. CNNs and DCNNs are also suitable for very large-scale integration (VLSI) implementations of associative memories. It is also expected that they can be applied to association memories by storing patterns as periodic limit cycles. According to theorem 1 and corollary 1, the $n$-neuron DCNN model, equation 2.1, can store up to $2^n$ patterns in locally exponentially attractive limit cycles or equilibria, which can be retrieved when the input vector satisfies condition 3.1. This implies that the external stimuli also play a major role in encoding and decoding patterns in DCNN associative memories, in contrast with the zero input vector in the bidirectional associative memories and the autoassociative memories based on the Hopfield network.

4 Locally Exponentially Attractive Periodicity in a Designated Region

As the limit cycles are stimulus driven (nonautonomous), some information can be encoded in the phases of the oscillating states $x_i$ relative to the inputs $u_i$. Hence, it is necessary to find some conditions on the inputs $u_i$, when the periodic orbit $x(t)$ is desired to be located in a designated region. In this section, we give the conditions that allow a periodic orbit to be locally exponentially attractive and located in any designated region.

Theorem 2. If $\forall t \geq t_0$,

$$u_i(t) < -1 + \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}) - \sum_{j \in N_3} (|a_{ij}| + |b_{ij}|), \quad i \in N_1,$$

(4.1)

$$u_i(t) > 1 + \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}) + \sum_{j \in N_3} (|a_{ij}| + |b_{ij}|), \quad i \in N_2,$$

(4.2)
\begin{align}
    u_i(t) &< 1 - a_{ii} - \sum_{j \in N_3, j \neq i} |a_{ij}| - \sum_{j \in N_3} |b_{ij}| + \sum_{j \in N_1} (a_{ij} + b_{ij}) \\
    &\quad - \sum_{j \in N_2} (a_{ij} + b_{ij}), \quad i \in N_3, \tag{4.3}
\end{align}

\begin{align}
    u_i(t) &> a_{ii} - 1 + \sum_{j \in N_3, j \neq i} |a_{ij}| + \sum_{j \in N_3} |b_{ij}| + \sum_{j \in N_1} (a_{ij} + b_{ij}) \\
    &\quad - \sum_{j \in N_2} (a_{ij} + b_{ij}), \quad i \in N_3, \quad \tag{4.4}
\end{align}

and \( \forall i \in \{1, 2, \ldots, n\}, \ j \in N_3, \ \tau_{ij}(t) = \tau_{ij}(t + \omega) \), then DCNN, equation 2.1, has only one limit cycle located in \( D_1 \), which is locally exponentially attractive in \( D_1 \).

**Proof.** If \( \forall s \in [t_0 - \tau, t] \), \( x(s) \in D_1 \), then from equation 2.1, \( \forall s \in [t_0, t] \), \( \forall i = 1, 2, \ldots, n \),

\[
\frac{dx_i(s)}{dt} = -x_i(s) - \sum_{j \in N_1} (a_{ij} + b_{ij}) + \sum_{j \in N_2} (a_{ij} + b_{ij}) + a_{ii}x_i(s) \\
+ \sum_{j \in N_3} b_{ij}x_j(s - \tau_{ij}(s)) + u_i(s). \tag{4.5}
\]

When \( i \in N_1 \) and \( x_i(t) = -1 \), from equations 4.1 and 4.5,

\[
\frac{dx_i(t)}{dt} \leq 1 - \sum_{j \in N_1} (a_{ij} + b_{ij}) + \sum_{j \in N_2} (a_{ij} + b_{ij}) \\
+ \sum_{j \in N_3} (|a_{ij}| + |b_{ij}|) + u_i(t) < 0. \tag{4.6}
\]

When \( i \in N_2 \) and \( x_i(t) = 1 \), from equations 4.2 and 4.5,

\[
\frac{dx_i(t)}{dt} \geq -1 - \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_3} (|a_{ij}| + |b_{ij}|) \\
+ \sum_{j \in N_2} (a_{ij} + b_{ij}) + u_i(t) > 0. \tag{4.7}
\]

When \( i \in N_3 \) and \( x_i(t) = 1 \), from equations 4.3 and 4.5,

\[
\frac{dx_i(t)}{dt} \leq -1 - \sum_{j \in N_1} (a_{ij} + b_{ij}) + \sum_{j \in N_2} (a_{ij} + b_{ij}) + a_{ii} \\
+ \sum_{j \in N_3, j \neq i} |a_{ij}| + \sum_{j \in N_3} |b_{ij}| + u_i(t) < 0. \tag{4.8}
\]
When \( i \in N_3 \) and \( x_i(t) = -1 \), from equations 4.4 and 4.5,

\[
\frac{dx_i(t)}{dt} \geq 1 - \sum_{j \in N_1} (a_{ij} + b_{ij}) + \sum_{j \in N_2} (a_{ij} + b_{ij}) - a_{ii} - \sum_{j \in N_3, j \neq i} |a_{ij}| - \sum_{j \in N_3} |b_{ij}| \exp\{\vartheta \tau\} + \vartheta \geq 0, \quad i \in N_3.
\]

(4.13)

Equations 4.6 to 4.9 imply that if \( \forall s \in [t_0 - \tau, t_0], \phi(s) \in D_1 \), then \( x(t; t_0, \phi) \) will keep in \( D_1 \), and \( D_1 \) is an invariant set of DCNN (see equation 2.1). So \( \forall t \geq t_0 - \tau, x(t) \in D_1 \). Hence, \( \forall t \geq t_0, \) DCNN, equation 2.1, can be rewritten as

\[
\frac{dx_i(t)}{dt} = -x_i(t) - \sum_{j \in N_1} (a_{ij} + b_{ij}) + \sum_{j \in N_2} a_{ij} x_j(t) + \sum_{j \in N_3} b_{ij} x_j(t - \tau_j(t)) + \sum_{j \in N_2} (a_{ij} + b_{ij}) + u_i(t), \quad i = 1, 2, \ldots, n.
\]

(4.10)

Let \( x(t; t_0, \phi) \) and \( x(t; t_0, \varphi) \) be two states of DCNN (equation 4.10) with initial conditions \( (t_0, \phi) \) and \( (t_0, \varphi) \), where \( \phi, \varphi \in C([t_0 - \tau, t_0], D_1) \). From equation 4.10, \( \forall i = 1, 2, \ldots, n; \forall t \geq t_0, \)

\[
\frac{d(x_i(t; t_0, \phi) - x_i(t; t_0, \varphi))}{dt} = -(x_i(t; t_0, \phi) - x_i(t; t_0, \varphi)) + \sum_{j \in N_3} (a_{ij} x_j(t; t_0, \phi) - x_j(t; t_0, \phi)) + b_{ij} (x_j(t - \tau_j(t); t_0, \phi) - x_j(t - \tau_j(t); t_0, \varphi)).
\]

(4.11)

Let \( y_i(t) = x_i(t; t_0, \phi) - x_i(t; t_0, \varphi) \). Then from equation 4.11, for \( i = 1, \ldots, n; \forall t \geq t_0, \)

\[
\frac{dy_i(t)}{dt} = -y_i(t) + \sum_{j \in N_3} a_{ij} y_j(t) + \sum_{j \in N_3} b_{ij} y_j(t - \tau_j(t)).
\]

(4.12)

From equations 4.3 and 4.4, for \( i \in N_3, \quad a_{ii} + |b_{ii}| + \sum_{j \in N_3, j \neq i} (|a_{ij}| + |b_{ij}|) + |\sum_{j \in N_3} (a_{ij} + b_{ij}) - \sum_{j \in N_3} (a_{ij} + b_{ij}) - u_i(t)| < 1. \) Hence, there exists \( \bar{\vartheta} > 0 \) such that

\[
(1 - a_{ii}) - \left( \sum_{j \in N_3, j \neq i} |a_{ij}| + \sum_{j \in N_3} |b_{ij}| \exp\{\bar{\vartheta} \tau\} \right) + \bar{\vartheta} \geq 0, \quad i \in N_3.
\]

(4.13)
Consider a subsystem of equation 4.12:

\[
\frac{dy_i(t)}{dt} = -y_i(t) + \sum_{j \in N_3} a_{ij} y_j(t) + \sum_{j \in N_3} b_{ij}(t - \tau_{ij}(t)), \ t \geq t_0, \ i \in N_3.
\]  

(4.14)

Denote \( ||\bar{y}||_{t_0} = \max_{0 \leq s \leq t_0} ||y(s)|| \). Then for \( i \in N_3, \ \forall t \geq t_0, \ |y_i(t)| \leq ||\bar{y}||_{t_0} \exp(-\vartheta(t - t_0)). \) Otherwise, one of the following two cases holds:

Case i. There exist \( t_2 > t_1 \geq t_0, k \in N_3 \), sufficiently small \( \varepsilon_1 > 0 \) such that \( y_k(t_1) - ||\bar{y}||_{t_0} \exp(-\vartheta(t_1 - t_0)) = 0, \ y_k(t_2) - ||\bar{y}||_{t_0} \exp(-\vartheta(t_2 - t_0)) = \varepsilon_1, \) and when \( s \in [t_0 - \tau, t_2] \), for all \( i \in N_3, \ |y_i(s)| - ||\bar{y}||_{t_0} \exp(-\vartheta(s - t_0)) \leq \varepsilon_1, \) and

\[
\frac{dy_k(t)}{dt} \bigg|_{t=t_1} + \vartheta ||\bar{y}||_{t_0} \exp(-\vartheta(t_1 - t_0)) \geq 0,
\]

\[
\frac{dy_k(t)}{dt} \bigg|_{t=t_2} + \vartheta ||\bar{y}||_{t_0} \exp(-\vartheta(t_2 - t_0)) > 0.
\]  

(4.15)

Case ii. There exist \( t_4 > t_3 \geq t_0, j \in N_3 \), sufficiently small \( \varepsilon_2 > 0 \) such that \( y_j(t_3) + ||\bar{y}||_{t_0} \exp(-\vartheta(t_3 - t_0)) = 0, \ y_j(t_4) + ||\bar{y}||_{t_0} \exp(-\vartheta(t_4 - t_0)) = -\varepsilon_2, \) and when \( s \in [t_0 - \tau, t_4] \), for all \( i \in N_3, \ |y_i(s)| - ||\bar{y}||_{t_0} \exp(-\vartheta(s - t_0)) \geq -\varepsilon_2, \) and

\[
\frac{dy_j(t)}{dt} \bigg|_{t=t_3} - \vartheta ||\bar{y}||_{t_0} \exp(-\vartheta(t_3 - t_0)) \leq 0,
\]

\[
\frac{dy_j(t)}{dt} \bigg|_{t=t_4} - \vartheta ||\bar{y}||_{t_0} \exp(-\vartheta(t_4 - t_0)) < 0.
\]  

(4.16)

It follows from equations 4.13 and 4.14 that for \( k \in N_3, \)

\[
\frac{dy_k(t)}{dt} \bigg|_{t=t_2} = -y_k(t_2) + \sum_{j \in N_3} (a_{kj} y_j(t_2) + b_{kj} y_j(t_2 - \tau_{kj}(t_2)))
\]

\[
\leq ||\bar{y}||_{t_0} \exp(-\vartheta(t_2 - t_0)) \left[ -1 + a_{kk} + \left( \sum_{j \in N_3, j \neq k} |a_{kj}| \right) \right.
\]

\[
+ \sum_{j \in N_3} |b_{kj}| \exp(\vartheta \tau) \left. \right] + \vartheta ||\bar{y}||_{t_0} \exp(-\vartheta(t_2 - t_0))
\]

\[
+ \varepsilon_1 \left[ -1 + a_{kk} + \left( \sum_{j \in N_3, j \neq k} |a_{kj}| + \sum_{j \in N_3} |b_{kj}| \right) \right]
\]

\[
\leq -\vartheta ||\bar{y}||_{t_0} \exp(-\vartheta(t_2 - t_0)).
\]
This contradicts equation 4.15. Similarly, it follows from equations 4.13 and 4.14 that
\[ \frac{dy_j(t)}{dt} \bigg|_{t=t_0} \geq \vartheta \| \bar{y} \|_{l_0} \exp(-\vartheta(t_0 - t_0)). \]

This contradicts equation 4.16. The two contradictions show that for \( i \in N_3, \forall t \geq t_0, \)
\[ |y_i(t)| \leq \| \bar{y} \|_{l_0} \exp(-\vartheta(t - t_0)). \]

Hence, according to lemma 2, there exists \( \bar{\vartheta} > 0 \) such that \( \forall i = 1, 2, \ldots, n, \forall t \geq t_0, \)
\[ |x_i(t; t_0, \phi) - x_i(t; t_0, \varphi)| \leq \| \phi, \varphi \|_{l_0} \exp(-\bar{\vartheta}(t - t_0)). \] (4.17)

Define \( x_\theta^{(i)}(\theta) = x(t + \theta; t_0, \phi), \theta \in [t_0 - \tau, t_0]. \) From equations 4.6 to 4.9, if \( \phi \in C([t_0 - \tau, t_0], D_1), \) then \( x_\theta^{(i)} \in C([t_0 - \tau, t_0], D_1). \) Define a mapping \( \bar{H} : C([t_0 - \tau, t_0], D_1) \rightarrow C([t_0 - \tau, t_0], D_1) \) by \( \bar{H}(\phi) = x_\theta^{(\omega)}, \) then \( \bar{H}(C([t_0 - \tau, t_0], D_1)) \subset C([t_0 - \tau, t_0], D_1), \) and \( \bar{H}^n(\phi) = x_\phi^{(\omega)} \).

Similar to the proof of theorem 1, there exists a periodic orbit \( x(t; t_0, \phi^*) \) of DCNN, equation 2.1, with period \( \omega \) such that \( \forall t \geq t_0, x(t; t_0, \phi^*) \in D_1 \) and all other states of DCNN, equation 2.1, with initial condition \( (t, \phi) (\phi \in C([t_0 - \tau, t_0], D_1)) \) converge to this periodic orbit exponentially as \( t \rightarrow +\infty. \) Hence, the isolated periodic orbit \( x(t; t_0, \phi^*) \) located in \( D_1 \) is locally exponentially attractive, and \( D_1 \) is a locally exponentially attractive set of \( x(t; t_0, \phi^*). \)

**Remark 3.** From equations 4.1 to 4.4, we can see that the input vector \( u(t) \) can control the locality of a limit cycle that represents a memory pattern in a designated region. Specifically, when condition 4.1 holds, the part in corresponding coordinate of the limit cycle is located in \((1, +\infty)\); when condition 4.2 holds, the part in corresponding coordinate of the limit cycle is located \((-\infty, -1)\); when conditions 4.3 and 4.4 hold, the part in corresponding coordinate of the limit cycle is located \([-1, 1]\).

When \( N_3 \) is empty, we have the following corollary:

**Corollary 2.** Let \( N_1 \cup N_2 = \{1, 2, \ldots, n\}, \) and \( N_1 \cap N_2 \) be empty. If \( \forall t \geq t_0, \)
\[ u_i(t) < \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}) - 1, \quad i \in N_1, \] (4.18)
\[ u_i(t) > \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}) + 1, \quad i \in N_2, \] (4.19)
then DCNN, equation 2.1, has exactly one limit cycle located in \( D_2 \), and such a limit cycle is locally exponentially attractive.

**Proof.** Let \( N_3 \) in theorem 2 be an empty set. According to theorem 2, corollary 2 holds.

When the periodic external input \( u(t) \) degenerates into a constant, we have the following corollary.

**Corollary 3.** If \( \forall t \geq t_0, u_i(t) \equiv u_i \) (constant), and

\[
\begin{align*}
    u_i &< -1 + \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}) - \sum_{j \in N_3} (|a_{ij}| + |b_{ij}|), \quad i \in N_1, \\
    u_i &> 1 + \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}) + \sum_{j \in N_3} (|a_{ij}| + |b_{ij}|), \quad i \in N_2, \\
    u_i &< 1 - a_{ii} - \sum_{j \in N_3, j \neq i} |a_{ij}| - \sum_{j \in N_2} |b_{ij}| + \sum_{j \in N_3} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}), \quad i \in N_3, \\
    u_i &> a_{ii} - 1 + \sum_{j \in N_3, j \neq i} |a_{ij}| + \sum_{j \in N_2} |b_{ij}| + \sum_{j \in N_3} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij}), \quad i \in N_3,
\end{align*}
\]

and \( \forall i \in \{1, 2, \ldots, n\}, \ j \in N_3, \ \tau_{ij}(t) \equiv \tau_{ij} \) (constant), then DCNN, equation 2.1, has only one equilibrium point located in \( D_1 \), which is locally exponentially stable.

**Proof.** Since \( u_i(t) \equiv u_i \) (constant), for arbitrary constant \( \nu \in \mathbb{R} \), \( u_i(t + \nu) \equiv u_i \equiv u_i(t) \). According to theorem 2, DCNN, equation 2.1, has only one limit cycle located in \( D_1 \), which is locally exponentially attractive in \( D_1 \). The arbitrariness of constant \( \nu \) implies that such a limit cycle is a fixed point. Hence, DCNN, equation 2.1, has only one equilibrium point located in \( D_1 \), which is locally exponentially stable.

## 5 Globally Exponentially Attractive Periodicity in a Designated Region

In order to obtain optimal spatiotemporal coding in the periodic orbit and reduce computational time, it is desirable for a neural network to be globally exponentially attractive to periodic orbit in a designated region. In this section, we give some conditions that allow a periodic orbit to be globally exponentially attractive and to be located in any designated region.

**Theorem 3.** If \( \forall t \geq t_0 \),

\[
u_i(t) < -1 - \sum_{j=1}^{N} (|a_{ij}| + |b_{ij}|), \quad i \in N_1,
\] (5.1)
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\[ u_i(t) > 1 + \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|), \ i \in N_2, \quad (5.2) \]

\[ |u_i(t)| < 1 - a_{ii} - \sum_{j=1, j \neq i}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|, \ i \in N_3, \quad (5.3) \]

and \( \forall i \in \{1, 2, \ldots, n\}, j \in N_3, \ \tau_{ij}(t) = \tau_{ij}(t + \omega) \), then DCNN, equation 2.1, has a unique limit cycle located in \( D_1 \), and such a limit cycle is globally exponentially attractive.

**Proof.** When \( i \in N_1 \) and \( x_i(t) \geq -1 \), from equations 2.1 and 5.1,

\[ \frac{dx_i(t)}{dt} \leq 1 + \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|) + u_i(t) < 0. \quad (5.4) \]

When \( i \in N_2 \) and \( x_i(t) \leq 1 \), from equations 2.1 and 5.2,

\[ \frac{dx_i(t)}{dt} \geq -1 - \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|) + u_i(t) > 0. \quad (5.5) \]

When \( i \in N_3 \) and \( x_i(t) \leq -1 \), from equations 2.1 and 5.3,

\[ \frac{dx_i(t)}{dt} \geq 1 - a_{ii} - \sum_{j=1, j \neq i}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}| + u_i(t) > 0. \quad (5.6) \]

When \( i \in N_3 \) and \( x_i(t) \geq 1 \), from equations 2.1 and 5.3,

\[ \frac{dx_i(t)}{dt} \leq -1 + a_{ii} + \sum_{j=1, j \neq i}^{n} |a_{ij}| + \sum_{j=1}^{n} |b_{ij}| + u_i(t) < 0. \quad (5.7) \]

Equations 5.4 to 5.7 imply that \( x(t; t_0, \phi) \) will go into and keep in \( D_1 \), where \( \phi \in C([t_0 - \tau, t_0], \mathbb{R}^n) \). So there exists \( T > 0 \) such that \( \forall t \geq T, x(t) \in D_1 \). Hence, \( \forall t \geq T + \tau \), DCNN, equation 2.1, can be rewritten as

\[
\frac{dx_i(t)}{dt} = -x_i(t) - \sum_{j \in N_1} (a_{ij} + b_{ij}) + \sum_{j \in N_3} a_{ij}x_j(t) + \sum_{j \in N_3} b_{ij}x_j(t - \tau_{ij}(t)) \\
+ \sum_{j \in N_2} (a_{ij} + b_{ij}) + u_i(t), \quad i = 1, 2, \ldots, n.
\]
Similar to the proof of theorem 2, DCNN, equation 2.1, has a unique limit cycle located in $D_1$, and such a limit cycle is globally exponentially attractive.

**Remark 4.** By comparison, we can see that if conditions 5.1 to 5.3 hold, then conditions 4.1 to 4.4 also hold. But not vice versa, as will be shown in examples 2 and 3. In other words, the conditions in theorem 3 are stronger than those in theorem 2.

When $N_1 \cup N_2$ is empty, we have the following corollary:

**Corollary 4.** If $\forall i, j \in \{1, 2, \ldots, n\}$, $\tau_{ij}(t) = \tau_{ij}(t + \omega)$, and $\forall t \geq t_0$,

$$|u_i(t)| < 1 - a_{ii} - \sum_{j=1, j \neq i}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|,$$

then the DCNN, equation 2.1, has a unique limit cycle located in $[-1, 1]^n$, which is globally exponentially attractive.

**Proof.** Choose $N_3 = \{1, 2, \ldots, n\}$ in theorem 3. According to theorem 3, the corollary holds.

When $N_3$ is empty, we have the following corollary:

**Corollary 5.** Let $N_3$ be empty. If

$$u_i(t) < -1 - \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|), \quad i \in N_1, \quad (5.8)$$
$$u_i(t) > 1 + \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|), \quad i \in N_2, \quad (5.9)$$

then DCNN, equation 2.1, has a unique limit cycle located in $D_2$. Moreover, such a limit cycle is globally exponentially attractive.

**Proof.** Since $N_3$ is empty, according to theorem 3, corollary 5 holds.

**Remark 5.** Since $-1 - \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|) \leq -1 + \sum_{j \in N_1} (a_{ij} + b_{ij}) - \sum_{j \in N_2} (a_{ij} + b_{ij})$, if condition 5.8 holds, then condition 4.18 also holds, but not vice versa. Similarly, if condition 5.9 holds, then condition 4.19 also holds, but not vice versa. This implies that the conditions in corollary 5 are stronger than those in corollary 2. In addition, corollary 5 shows that a DCNN has a globally exponentially attractive limit cycle if its periodic external stimulus is sufficiently strong.
When the periodic external input \( u(t) \) degenerates into a constant vector, we have the following corollary:

**Corollary 6.** If \( \forall t \geq t_0 \), \( u_i(t) \equiv u_i \) (constant), and

\[
\begin{align*}
  u_i < -1 - \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|), & \quad i \in N_1, \\
  u_i > 1 + \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|), & \quad i \in N_2, \\
  |u_i| < 1 - a_{ii} - \sum_{j=1, j \neq i}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|, & \quad i \in N_3,
\end{align*}
\]

and \( \forall i \in \{1, 2, \ldots, n\}, j \in N_3 \), \( \tau_{ij}(t) \equiv \tau_{ij} \) (constant), then DCNN, equation 2.1, has a unique equilibrium point located in \( D_1 \) and is globally exponentially stable at such an equilibrium point.

**Proof.** Since \( u_i(t) \equiv u_i \) (constant), for an arbitrary constant \( \nu \in \mathbb{R} \), \( u_i(t + \nu) \equiv u_i \). According to theorem 3, DCNN, equation 2.1, has a unique limit cycle located in \( D_1 \), which is globally exponentially attractive. The arbitrariness of constant \( \nu \) implies that such a limit cycle is a fixed point. Hence, DCNN, equation 2.1, has a unique equilibrium point located in \( D_1 \), and such an equilibrium point is globally exponentially stable.

### 6 Illustrative Examples

In this section, we give three numerical examples to illustrate the new results.

#### 6.1 Example 1.

Consider a CNN, where

\[
A = \begin{pmatrix}
2 & 0.2 & 0.2 \\
0.2 & 2.4 & 0.2 \\
0.4 & 0.6 & 2.4
\end{pmatrix}
\quad
u(t) = \begin{pmatrix}
0.5 \sin(t) \\
-0.6 \cos(t) \\
-0.2(\sin(t) + \cos(t))
\end{pmatrix}.
\]

According to theorem 1, this CNN has \( 2^3 = 8 \) limit cycles, which are locally exponentially attractive. Simulation results with 136 random initial states are depicted in Figures 1 to 3.
Figure 1: Transient behavior of $x_1, x_2, x_3$ in Example 1.

Figure 2: Transient behavior of $(x_1, x_3)$ in Example 1.
6.2 Example 2. Consider a CNN, where

\[
A = \begin{pmatrix}
2 & 0.2 & 0.2 \\
0.5 & 0.4 & 0.5 \\
-1.4 & 0.4 & 0.8
\end{pmatrix}
\quad u(t) = \begin{pmatrix}
0.5 \sin(t) \\
0.5 \cos(t) \\
0.5(\sin(t) + \cos(t))
\end{pmatrix}.
\]

Choose \( N_1 = \{1\}, N_2 = \{3\}, N_3 = \{2\} \). Since \( u_1(t) < -1 + a_{11} - a_{13} - |a_{12}|; a_{22} + |a_{21} - a_{23} - u_2(t)| < 1; u_3(t) > 1 + a_{31} - a_{33} + |a_{32}| \), according to theorem 2, this CNN has a limit cycle located in \( D'_1 = \{x| x_1 < -1, |x_2| \leq 1, x_3 > 1\} \), which is locally exponentially attractive in \( D'_1 \).

Choose \( N_1 = \{3\}, N_2 = \{1\}, N_3 = \{2\} \). Since \( u_1(t) > 1 + a_{13} - a_{11} + |a_{12}|, a_{22} + |a_{21} - a_{23} - u_2(t)| < 1, u_3(t) < -1 + a_{33} - a_{31} - |a_{32}| \), according to theorem 2, this CNN has a limit cycle located in \( D''_1 = \{x| x_3 < -1, |x_2| \leq 1, x_1 > 1\} \), which is locally exponentially attractive in \( D''_1 \). However, since \( 0.5 \sin(t) > 1 + (2 + 0.2 + 0.2) = 3.4 \) does not hold (i.e., condition 5.1 does not hold), it does not satisfy the conditions in theorem 3. Simulation results with 136 random initial states are depicted in Figures 4 and 5.

Figure 3: Transient behavior of \((x_1, x_2, x_3)\) in Example 1.
Figure 4: Transient behavior of $x_1$, $x_2$, $x_3$ in Example 2.

Figure 5: Transient behavior of $(x_1, x_2, x_3)$ in Example 2.
6.3 Example 3. Consider a CNN, where

\[
A = \begin{pmatrix} 0.2 & 0.2 & 0.2 \\ 0.2 & -2 & 0.6 \\ 0.2 & 0.2 & 0.2 \end{pmatrix} \quad u(t) = \begin{pmatrix} 0.8 \sin(t) - 2.6 \\ 0.8 \cos(t) \\ 0.8(\sin(t) + \cos(t)) + 2.8 \end{pmatrix}.
\]

Choose \( N_1 = \{1\} \), \( N_2 = \{3\} \), \( N_3 = \{2\} \). Since \( u_1(t) < -1 - \sum_{j=1}^{3} |a_{1j}| \), \( |u_2(t)| < 1 - a_{22} - \sum_{j=1, j \neq 2}^{3} |a_{2j}| \), \( u_3(t) > 1 + \sum_{j=1}^{3} |a_{3j}| \), according to theorem 3, this CNN has a limit cycle located in \( D'_1 = \{x| x_1 < -1, |x_2| \leq 1, x_3 > 1\} \), which is globally exponentially attractive. Since \( u_1(t) < -1 + a_{11} - |a_{12}| - a_{13} \); \( u_2(t) < 1 - a_{22} + a_{21} - a_{23} \); \( u_2(t) > -1 + a_{22} + a_{21} - a_{23} \); \( u_3(t) > 1 + a_{31} + |a_{32}| - a_{33} \), conditions 4.1 to 4.4 also hold. According to theorem 2, this CNN has a limit cycle located in \( D'_1 \), which is also locally exponentially attractive. However, since \( a_{11} > 0, a_{33} > 0 \) the conditions in Yi et al. (2003) cannot be used to ascertain the complete stability of this CNN. Simulation results with 136 random initial states are depicted in Figures 6 and 7.
7 Concluding Remarks

Rhythmicity represents one of most striking manifestations of dynamic behaviors in biological systems. CNNs and DCNNs, which have been shown to be capable of operating in a pacemaker or pattern generator mode, are studied here as oscillatory mechanisms in response to periodic external stimuli. Some information can be encoded in the oscillating activation states relative to external inputs, and these relative phases change as a function of the chosen limit cycle. In this article, we show that the number of locally exponentially attractive periodic orbits located in saturation regions in a DCNN is exponential of the number of the neurons. In view of the fact that neural information is often desired to be encoded in a designated region, we also give conditions to allow a globally exponentially attractive periodic orbit located in any designated region. The theoretical results are supplemented by simulation results in three illustrative examples.

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