Global Uniform Asymptotic Stability of Memristor-based Recurrent Neural Networks with Time Delays

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Abstract—Memristor is a newly prototyped nonlinear circuit device. Its value is not unique and changes according to the value of the magnitude and polarity of the voltage applied to it. In this paper, a simplified mathematical model is proposed to characterize the pinched hysteretic feature of the memristor, a memristor-based recurrent neural network model is given, and its global stability is studied. Using differential inclusion, two sufficient conditions for the global uniform asymptotic stability of memristor-based recurrent neural networks are obtained.

I. INTRODUCTION

In 1971, Leon O. Chua [1] reasoned from physical symmetry arguments that, besides the resistor, capacitor and inductor, there should be a fourth circuit element which he called a memristor (as a contraction of memory and resistor). Although he showed that such an element has many interesting and valuable features, no much attention was paid to his theory because no one could ever build one until nearly 40 years later. In 2008, a group of scientists from Hewlett-Packard Lab announced that they had built a prototype of the memristor based on nanotechnology [2][3].

Unlike the other three fundamental circuit elements, the memristor is a nonlinear one and its value is not unique. Chua theoretically showed that the value of the memristor, called memristance, is the function of electric charge $q$ given as follows:

$$M(q) = \frac{d\phi}{dq}$$

where $\phi$ denotes the magnetic flux. Its current-voltage characteristic is similar to that of a Lissajous pattern in Figure 1 and is demonstrated by experiments from the scientists at the HP Lab (see Fig. 3d in [2]).

The memristor is a contraction of “memory resistor” due to its function: to memorize its history. A memristor is a two-terminal passive device whose value (i.e., memristance) depends on the magnitude and polarity of the voltage applied to it and the length of time that the voltage has been applied. When the voltage is turned off, the memristor remembers its most recent value until next time it is turned on. From Fig. 1, we can see that the memristor exhibits the feature of pinched hysteresis, which means that a lag occurs between the application and the removal of a field and its subsequent effect, just as the neurons in the human brain have. Because of this feature, broad potential applications of memristors have been identified [4][5][6], one of which is to apply this device to build a new model of neural networks to emulate the human brain.

As is well known, neural networks can be implemented by VLSI circuits. For example, the Hopfield neural network model can be implemented in a circuit where the self-feedback connection weights and the connection weights are implemented by resistors. Suppose that we use memristors there instead of resistors, then we can build a new model where the parameters change according to its state; i.e., it is a state-dependent switching recurrent network. In the following sections, we will propose a simplified mathematical model of memristors. Based on this model, a recurrent neural network model with time delays is given and its global...
uniform asymptotic stability is analyzed.

II. PRELIMINARIES

A. Model Description

According to the feature of the memristor and the current-voltage characteristic given in Figure 1, we propose a simplified mathematical model of the memristance as follows:

\[ M(u(t)) = \begin{cases} M', & \dot{u}(t) > 0; \\ M'', & \dot{u}(t) < 0; \\ \text{unchanged}, & \dot{u}(t) = 0; \end{cases} \tag{1} \]

where \( u \) is the voltage applied to the memristor, \( \dot{u}(t) \) is the derivative of \( u \) with respect to time \( t \) and \( M' \geq M'' \). When \( \dot{u}(t) = 0 \), “unchanged” means that the memristance keeps the current value. From this representation we can see that the memristance switches between two values according to the voltage applied to the memristor. Since the connection weights can be implemented by using resistors, we can similarly utilize memristors to build a switching neural network.

Consider the following memristor-based recurrent neural network model with time delays

\[ \dot{z}_i(t) = -d_i z_i(t) + \sum_{j=1}^{n} a_{ij}(z_i - z_j) \hat{f}_j(z_j(t)) + \sum_{j=1}^{n} b_{ij}(z_i - z_j) \hat{g}_j(z_j(t - \tau(t))) + s_i, \tag{2} \]

where \( z_i(t) \) is the state variable of the \( i \)-th neuron, \( d_i \) is the \( j \)-th self-feedback connection weight, \( a_{ij}(z_i - z_j) \) and \( b_{ij}(z_i - z_j) \) are, respectively, connection weights and those associated with time delays, \( s_i \) is the \( i \)-th external input bias. \( \hat{f}_i(\cdot) \) and \( \hat{g}_i(\cdot) \) are the \( i \)-th activation functions and those associated with time delays satisfying the following assumption:

A1: \( \hat{f}_i(\cdot) \) and \( \hat{g}_i(\cdot) (i = 1, 2, \ldots, n) \) are Lipschitz continuous, that is, for any \( x, y \in R, x \neq y \), the following inequalities hold

\[ |\hat{f}_i(x) - \hat{f}_i(y)| \leq \mu_i |x - y|, \quad |\hat{g}_i(x) - \hat{g}_i(y)| \leq \lambda_i |x - y|. \]

Since the connection weights are implemented by using memristors, according to Eq. (1), the value of \( a_{ij} \) and \( b_{ij} \) \((i \neq j)\) are functions of \( z_i \) and \( z_j \) defined as

\[ w_{ij}(z_i - z_j) = \begin{cases} w_{ij}', & z_i - z_j > 0; \\ w_{ij}'', & z_i - z_j < 0; \\ \text{unchanged}, & z_i - z_j = 0; \end{cases} \tag{3} \]

where \( w \) can be \( a \) or \( b \), \( a_{ii} \) and \( b_{ii} \) are constants. For convenience, we define that \( a_{ii}' = a_{ii}'' = a_{ii} \) and \( b_{ii}' = b_{ii}'' = b_{ii} \).

In the recent decade, numerous results on the global stability of recurrent neural networks have been reported, e.g., [7][8][9][10][11][12][13][14][15][16][17][18][19][20]. In system (2), since \( a_{ij}(\cdot) \) and \( b_{ij}(\cdot) \) are discontinuous, the classical definition of the solution for differential equations cannot apply here. To handle this problem, A. F. Filippov [21] developed a solution concept for the differential equation with a discontinuous right-hand side. Based on this definition, a differential equation with a discontinuous right-hand side has the same solution set as a certain differential inclusion. Hence, to analyze the stability of such differential equations we can turn to study relevant differential inclusions. Next we will introduce some useful definitions and lemmas about differential inclusions.

B. Definitions and Lemmas

Consider a system of ordinary differential equations of the form

\[ \dot{x} = h(t, x), \tag{4} \]

where \( h: R \times R^n \rightarrow R^n \) is discontinuous. The Filippov solution of (4) is given as follows:

Definition 1: A function \( x(\cdot) \) is called a solution of (4) on \([t_0, t_1]\) if \( x(\cdot) \) is absolutely continuous on \([t_0, t_1]\) and for almost all \( t \in [t_0, t_1] \),

\[ \dot{x} \in \tilde{h}(t, x), \tag{5} \]

where

\[ \tilde{h}(t, x) = \text{co} \{ v : \exists \{x_i\} \text{ with } x_i \rightarrow x \text{ such that } x_i \notin N_0' \cup N \text{ and } v = \lim h(t, x_i) \}. \tag{6} \]

According to this definition, the switching system

\[ \dot{x} = \varphi_\sigma(x, t), \quad \sigma \in \Omega, \tag{8} \]

has the same solution set as the following differential inclusion

\[ \dot{x} \in \text{co} \{ \varphi_\sigma(x, t) \}, \tag{9} \]

where \( \sigma: [0, +\infty) \rightarrow \Omega \) is an arbitrary switching signal, \( \Omega \) is the index set, \( \varphi(t, x) \) is locally bounded and \( \text{co} \{ \varphi_\sigma(x) \} \) denotes the convex hull of \( \varphi_\sigma(x) \). Then to analyze the stability of differential equations (8), we can turn to study the relevant differential inclusion (9).

First, to guarantee the existence of the solution of the differential inclusion, we need the following lemmas:

Lemma 1 [23]: Let \( \tilde{h}(t, x) \) be a set-valued map defined in (6) or (7). If \( \tilde{h}(t, x) \) is locally bounded and Lebesgue measurable with respect to \((t, x) \in [0, +\infty) \times R^n \), then \( \tilde{h}(t, x) \) satisfies:

A2: \( \tilde{h}(t, x) \) is a nonempty, compact, convex subset of \( R^n \) for any \( t \geq 0 \) and \( x \in R^n \);

A3: \( \tilde{h}(t, x) \), as a set-valued map of \( x \), is upper semi-continuous for any \( t \geq 0 \);
A4: \( \tilde{h}(t, x) \), as a set-valued map of \( x \), is Lebesgue measurable for any \( x \in \mathbb{R}^n \); 
A5: \( \tilde{h}(t, x) \) is locally bounded.

**Lemma 2** [23]: Let \( \tilde{h}(t, x) \) be a set-valued map. If \( \tilde{h}(t, x) \) fulfills A2, A3, A4 and A5, then for any \( (t, x) \in [0, +\infty) \times \mathbb{R}^n \), there exists an interval \( I \) and at least a solution \( x(t) : I \rightarrow \mathbb{R}^n \) of (5) such that \( t_0 \in I \) and \( x(t_0) = x_0 \).

Suppose that the solution \( x(t) \) of the differential inclusion (5) exists such that \( x(t_0) = x_0 \) and the origin is an equilibrium point for (5); i.e., \( 0 \in \tilde{h}(t, 0) \) for a.e. \( t \geq 0 \).

The definition of the global uniform asymptotic stability is given followed by the sufficient condition of the stability.

**Definition 2**: The differential inclusion (5) is said to be globally uniformly asymptotically stable, if its equilibrium point is uniformly stable and globally uniformly attractive and all the solutions are uniformly bounded. Precisely, there exist two functions \( m : (0, +\infty) \rightarrow (0, +\infty) \) and \( T : (0, +\infty) \times (0, +\infty) \rightarrow (0, +\infty) \) such that

1) for any \( r > 0 \), any \( (t_0, x_0) \in (0, +\infty) \times \mathbb{R}^n \) and any solution \( x(t) \)

\[ ||x_0|| \leq r \Rightarrow ||x(t)|| < m(r), \quad \forall t \geq t_0; \]

2) \( \lim_{r \to 0^+} m(r) = 0; \)

3) for any \( r > 0 \), any \( \varepsilon > 0 \), any \( (t_0, x_0) \in (0, +\infty) \times \mathbb{R}^n \),

\[ ||x_0|| \leq r \Rightarrow ||x(t)|| < \varepsilon, \quad \forall t \geq t_0 + T(r, \varepsilon). \]

**Remark 1**: In Definition 2, 1) means that all the solutions of differential inclusion (5) are uniformly bounded, 2) together with 2) means that (5) is uniformly asymptotically stable, and 3) means that (5) is globally uniformly attractive. The global uniform asymptotic stability is obviously stronger than global asymptotic stability. Furthermore, if a differential inclusion is exponentially stable, it is also uniformly attractive. Hence the global uniform asymptotic stability is weaker than global exponential stability.

**Lemma 3** [23]: Let \( \tilde{h} : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a set-valued map such that the existence of (5) is insured. Assume that there exists a strict Lyapunov function in the large \( V \); i.e., a function \( V = V(t, x) \) such that, for some functions \( a, b, c \in K_\infty^c \),

\[ a(||x||) \leq V(t, x) \leq b(||x||), \quad \forall t \in [0, +\infty), x \in \mathbb{R}^n; \]

\[ V(t_2, x(t_2)) - V(t_1, x(t_1)) \leq - \int_{t_1}^{t_2} c(||x(\tau)||)d\tau, \quad t_1 \leq t_2; \]

(11)

for any pair of time instants \( (t_1, t_2) \) and any solution \( x(t) : [t_1, t_2] \rightarrow \mathbb{R}^n \) of (5), then the equilibrium point is globally uniformly asymptotically stable for (5).

To simplify our proof, in the next section, we use the notion of \( M \)-matrix. Since there are many equivalent definitions of \( M \)-matrix, we choose one of them as follows:

**Definition 3** [24]: Let matrix \( A = (a_{ij})_{n \times n} \) has nonpositive off-diagonal elements. \( A \) is a nonsingular \( M \)-matrix if one of the following conditions holds:

1) All principal minors of \( A \) are positive;
2) \( A \) has all positive diagonal elements and there exists a positive diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \) such that \( A \Lambda \) is strictly diagonally dominant, that is

\[ a_{ii} \lambda_i > \sum_{j \neq i} |a_{ij}| \lambda_j, \quad i = 1, 2, \cdots, n; \]

which can be written as

\[ \sum_{j=1}^{n} a_{ij} \lambda_j > 0, \quad i = 1, 2, \cdots, n. \]

Throughout this paper, we denote \( ||v||_p \) as the vector \( p \)-norm of the vector \( v \) with \( 1 \leq p < \infty \), e.g.

\[ ||v||_1 = \sum_{i=1}^{n} |u_i|, \quad ||v||_2 = \sqrt{\sum_{i=1}^{n} u_i^2}. \]

### III. MAIN RESULTS

**System (2)** can be written in the following vector form

\[ \dot{z}(t) = P(z) = -Dz(t) + A(z)\tilde{f}(z(t)) + B(z)\tilde{g}(z(t - \tau(t))) + \varepsilon. \]

(12)

Since the activation functions \( f_i(x) \) and \( g_i(x) \) are Lipschitz continuous, they are locally bounded. Hence \( P(z) \) is also locally bounded. According to the analysis above, the memristor-based recurrent neural network (12) has the same solution set as the following differential inclusion:

\[ \dot{z} \in \text{co}\{P(z)\} = -Dz(t) + A\tilde{f}(z(t)) + B\tilde{g}(z(t - \tau(t))) + \varepsilon, \]

(13)

where \( D = \text{diag}(d_1, d_2, \cdots, d_n) \), \( A = (\xi_{ij}a_{ij} + (1 - \xi_{ij})a_{ij}^r)_{n \times n} \), \( B = (\xi_{ij}b_{ij} + (1 - \xi_{ij})b_{ij}^r)_{n \times n} \), \( \xi_{ij} \)'s are arbitrary constants such that \( 0 \leq \xi_{ij} \leq 1 \) and \( \xi_{ij} + \xi_{ji} = 1 \), \( \tilde{f}(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t))]^T \), \( \tilde{g}(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), \cdots, g_n(x_n(t))]^T \), \( s = (s_1, s_2, \cdots, s_n)^T \). Since \( P(z) \) is locally bounded, due to Lemma 1, \( \text{co}\{P(z)\} \) satisfies A2, A3, A4 and A5. Thus according to Lemma 2, the existence of the solution of (13) is ensured.

The differential inclusion (13) means that there exist some \( \xi_{ij} (i, j = 1, 2, \cdots, n) \) such that

\[ \dot{z}_i = -d_i z_i + \sum_{j=1}^{n} \xi_{ij} a_{ij} + (1 - \xi_{ij}) a_{ij}^r f_j(z_j(t)) \]

\[ + \sum_{j=1}^{n} \xi_{ij} b_{ij} + (1 - \xi_{ij}) b_{ij}^r g_j(z_j(t - \tau_j(t))) + \varepsilon_i. \]

(14)

For any \( \xi_{ij} \), assumption A1 ensures the existence of an equilibrium point \( z^* \) of (14). We shift the equilibrium point to the origin by the translation \( x = z - z^* \), which results in

\[ \dot{x}_i = -d_i x_i + \sum_{j=1}^{n} \xi_{ij} a_{ij} + (1 - \xi_{ij}) a_{ij}^r f_j(x_j(t)) \]

\[ + \sum_{j=1}^{n} \xi_{ij} b_{ij} + (1 - \xi_{ij}) b_{ij}^r g_j(x_j(t - \tau_j(t))). \]

(15)
Eq. (15) can be written in the form of differential inclusion as follows:
\[
\dot{x} \in -Dx(t) + Af(x(t)) + Bg(x(t - \tau)),
\]  
(16)
where \( D, A, B \) are defined the same as above, \( f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^T = \tilde{f}(z) - \tilde{f}(z^*) \) and \( g(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t))]^T = \tilde{g}(z) - \tilde{g}(z^*) \). Obviously, \( f_i(\cdot) \) and \( g_i(\cdot) \) satisfy assumption \( A_i \) and we can easily see that \( f_i(\cdot) \) and \( g_i(\cdot) \) also satisfy the following assumption:

\( A_i \): There exist positive constants \( \mu_i \) and \( \lambda_i \) such that for any \( x \in R \),
\[
|f_i(x)| \leq \mu_i|x|, \quad |g_i(x)| \leq \lambda_i|x|, \quad i = 1, 2, \ldots, n.
\]

**Theorem 1:** The memristor-based recurrent neural network (12) is globally uniformly asymptotically stable if
\[
D - |A|_\text{max}K - |B|_{\text{max}}L \text{ is an } M\text{-matrix, where } |A|_{\text{max}} = \max(|a_{ij}|_{\text{max}})_{n \times n} \text{ and } |B|_{\text{max}} = \max(|b_{ij}|_{\text{max}})_{n \times n} \text{, and we can easily see that } f_i(\cdot) \text{ and } g_i(\cdot) \text{ also satisfy the following assumption:}
\]

\[\begin{align*}
A_i: \text{ There exist positive constants } &\mu_i \text{ and } \lambda_i \text{ such that for any } x \in R, \\
|f_i(x)| &\leq \mu_i|x|, \quad |g_i(x)| \leq \lambda_i|x|, \quad i = 1, 2, \ldots, n.
\end{align*}\]

By (17), we have \( \omega > 0 \). Then integrating (18) from \( t_1 \) to \( t_2 \), we have
\[
V(x(t_2)) - V(x(t_1)) \leq - \int_{t_1}^{t_2} \omega||x(t)|| \, dt.
\]

Thus \( \omega \) is positive definite, (10) is satisfied. Therefore, by Lemma 3, differential inclusion (13) is globally uniformly asymptotically stable. Since (13) has the same solution set as (12), (12) is also globally uniformly asymptotically stable.

**Remark 2:** When \( b'_{ij} = b''_{ij} = 0 \) (i, j = 1, 2, \ldots, n), we have the following memristor-based recurrent neural network without time-delay:
\[
\dot{z}(t) = -Dz(t) + A(z)f(z(t)) + s.
\]
(19)

If \( D - |A|_{\text{max}}K \) is an \( M\text{-matrix, where } |A|_{\text{max}} = \max(|a_{ij}|_{\text{max}})_{n \times n}, \quad |a_{ij}|_{\text{max}} = \max\{|a'_{ij}|, |a''_{ij}|\}, \quad K = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n) \text{, then neural network (19) is globally uniformly asymptotically stable.}

**Remark 3:** Assume that for i, j = 1, 2, \ldots, n, \( a'_{ij} = a''_{ij} \), \( b'_{ij} = b''_{ij} \), then the switching recurrent neural network is degenerated to a conventional recurrent neural network as follows:
\[
\dot{z}(t) = -Dz(t) + Af(z(t)) + Bg(z(t - \tau(t))).
\]
(20)

Then using the method in Theorem 1, we can obtain the following global uniform asymptotic stability condition for system (20) as shown in [9]: The recurrent neural network (20) is globally uniformly asymptotically stable if \( D - |A|K - |B|L \)
is an $M$-matrix, where $D = \text{diag} (d_1, d_2, \ldots, d_n)$, $|A| = (|a_{ij}|)_{n \times n}$, $|B| = (|b_{ij}|)_{n \times n}$, $K = \text{diag} (\mu_1, \mu_2, \ldots, \mu_n)$, $L = \text{diag} (\lambda_1, \lambda_2, \ldots, \lambda_n)$. It implies that Theorem 1 is a generalization of the stability condition in [9].

Using the same method, we can obtain the following theorem.

**Theorem 2:** The memristor-based recurrent neural network (12) is globally uniformly asymptotically stable if there exist constants $\beta_i > 0 (i = 1, 2, \ldots, n)$ such that

$$2d_i \beta_i - \sum_{j=1}^n \beta_j |\mu_j| |a_{ij}|_{\text{max}} + \lambda_j |b_{ij}|_{\text{max}}$$

$$- \sum_{j=1}^n \beta_j |\mu_j| |a_{ji}|_{\text{max}} + \lambda_i |b_{ji}|_{\text{max}} > 0. \quad (21)$$

where $|a_{ij}|_{\text{max}} = \max\{|a_{ij}'|, |a_{ij}''|\}$ and $|b_{ij}|_{\text{max}} = \max\{|b_{ij}'|, |b_{ij}''|\}$.

**Proof:** Construct the following Lyapunov functional

$$V(t) = \frac{1}{2} \sum_{i=1}^n \beta_i x_i(t)^2 + \frac{1}{2} \sum_{i=1}^n \int_{t-\tau_i}^t \gamma_i x_i^2(s) ds,$$

where $\gamma_i = \sum_{j=1}^n \beta_j |\mu_j| |b_{ji}|_{\text{max}}$.

Deriving the upper Dini-derivative of $V(x(t))$ along (16), we have

$$D^+ V(t) = \sup \left\{ \sum_{i=1}^n \beta_i x_i(t) \left[ -d_i x_i(t) + \sum_{j=1}^n (\xi_{ij} a_{ij}' + (1-\xi_{ij}) a_{ij}'') f_j(x_j(t)) + \sum_{j=1}^n (\xi_{ij} b_{ij}' + (1-\xi_{ij}) b_{ij}'') g_j(x_j(t)) \right] \right\}
+ \frac{1}{2} \sum_{i=1}^n \left[ |\gamma_i x_i^2(t) - \gamma_i x_i^2(t - \tau_i) | \right]$$

$$\leq \sup \left\{ - \sum_{i=1}^n \beta_i d_i x_i^2(t) + \sum_{i=1}^n \sum_{j=1}^n \beta_j |\xi_{ij} a_{ij}'| \right\}
+ \frac{1}{2} \sum_{i=1}^n \left[ |\gamma_i x_i^2(t) - \gamma_i x_i^2(t - \tau_i) | \right]
+ \left[ 2d_i \beta_i - \sum_{j=1}^n \beta_i |\mu_j| |a_{ij}|_{\text{max}} + \lambda_j |b_{ij}|_{\text{max}} \right] x_i^2(t)
- \frac{1}{2} \omega \sum_{i=1}^n x_i^2(t) = -\frac{1}{2} \omega \|x\|^2,$$

where $\omega = 2d_i \beta_i - \sum_{j=1}^n \beta_i |\mu_j| |a_{ij}|_{\text{max}} + \lambda_j |b_{ij}|_{\text{max}} - \sum_{j=1}^n \beta_j |\mu_j| |a_{ji}|_{\text{max}} + \lambda_i |b_{ji}|_{\text{max}}$.

By (21), we have $\omega > 0$. Then integrating (22) from $t_1$ to $t_2$, we have

$$V(x(t_2)) - V(x(t_1)) \leq -\frac{1}{2} \int_{t_1}^{t_2} \omega \|x(t)\|^2 dt.$$

Thus (11) holds. Since $V(x(t))$ is positive definite, (10) is satisfied. Therefore, by Lemma 2, differential inclusion (13) is uniformly globally asymptotically stable. Since (13) has the same solution set as (12), (12) is also uniformly globally asymptotically stable.

**IV. ILLUSTRATIVE EXAMPLES**

In this section, we will give two numerical examples to demonstrate the results.

**Example 1:** Consider a memristor-based recurrent neural network with time-delays

$$\dot{x}(t) = -D x(t) + A(x) f(x(t)) + B(x) g(x(t - \tau)) + s, \quad (23)$$

where $D = \begin{pmatrix} 5 & 0 \\ 0 & 8 \end{pmatrix}$, $A(x) = \begin{pmatrix} 2 & a_{12} \\ a_{21} & -3 \end{pmatrix}$, $B(x) = \begin{pmatrix} -5 & b_{12} \\ b_{21} & -4 \end{pmatrix}$,
\begin{align*}
a_{12} &= \begin{cases} 
-1, & \dot{x}_1 > \dot{x}_2 \\
0.5, & \dot{x}_1 < \dot{x}_2
\end{cases}, \quad a_{21} = \begin{cases} 
3, & \dot{x}_2 > \dot{x}_1 \\
-4, & \dot{x}_2 < \dot{x}_1
\end{cases}, \\
b_{12} &= \begin{cases} 
2, & \dot{x}_1 > \dot{x}_2 \\
3, & \dot{x}_1 < \dot{x}_2
\end{cases}, \quad b_{21} = \begin{cases} 
-2, & \dot{x}_2 > \dot{x}_1 \\
1, & \dot{x}_2 < \dot{x}_1
\end{cases},
\end{align*}
and
\[f_i(x) = g_i(x) = \frac{1 - e^{-x}}{1 + e^{-x}}, \quad i = 1, 2.\]
Clearly, \(f_i\) and \(g_i\) are Lipschitz continuous with the Lipschitz constants \(\mu_i = \lambda_i = 1/2\). Note that
\[|A|_{\text{max}} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}, \quad |B|_{\text{max}} = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}, \quad D - |A|_{\text{max}}K - |B|_{\text{max}}L = \begin{pmatrix} 1.5 & -2 \\ -3 & 4.5 \end{pmatrix}.
\]
It can be checked that \(D - |A|_{\text{max}}K - |B|_{\text{max}}L\) is an \(M\)-matrix. As a result, according to Theorem 1, (23) is globally uniformly asymptotically stable.

Let \(s = (3, 2)^T\). The following four cases are given: Case 1 with the delay parameters \(\tau_1 = 0.5\) and \(\tau_2 = 0.2\), the initial state \(x_1(t) = 3\) for \(t \in [-\tau_1, 0]\) and \(x_2(t) = -2\) for \(t \in [-\tau_2, 0]\); Case 2 with the same delay parameters as in Case 1 and the initial state \(x_1(t) = -3\) for \(t \in [-\tau_1, 0]\) and \(x_2(t) = 4\) for \(t \in [-\tau_2, 0]\), Case 3 with the delay parameters \(\tau_1 = \tau_2 = 0.5\) and the initial state \(x(t) = (1, 2.5)^T\) for \(t \in [-0.5, 0]\), Case 4 with the same delay parameters as in Case 3 and the initial state \(x(t) = (-2, -0.5)^T\) for \(t \in [-0.5, 0]\). Fig. 2 depicts the time responses of state variables of the neural network for the above four cases.

If we change
\[D = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix},\]
then
\[D - |A|_{\text{max}}K - |B|_{\text{max}}L = \begin{pmatrix} 1.5 & -2 \\ -3 & 2.5 \end{pmatrix}\]
is not an \(M\)-matrix. Therefore, Theorem 1 can not be used to ascertain the stability. However, using Theorem 2, we can first assume that there exist positive constants \(\beta_1\) and \(\beta_2\) such that
\[2d_i\beta_i - \sum_{j=1}^{2}\beta_i(\mu_i|a_{ij}|_{\text{max}} + \lambda_j|b_{ij}|_{\text{max}}) - \sum_{j=1}^{2}\beta_j(\mu_i|a_{ji}|_{\text{max}} + \lambda_i|b_{ji}|_{\text{max}}) > 0, \quad i = 1, 2.
\]
Substituting \(d_1 = 5, d_2 = 6, |a_{12}|_{\text{max}} = 1, |a_{21}|_{\text{max}} = 4, |b_{12}|_{\text{max}} = 3, |b_{21}|_{\text{max}} = 2, \mu_i = \lambda_i = 1\) into the above inequalities results in
\[2\beta_2 < \beta_1 < 3\beta_2.
\]
Then the condition of Theorem 2 can be satisfied and (23) is thus globally uniformly asymptotically stable.

Let \(s = (4, -3)^T\). The following four cases are given: Case 1 with the delay parameters \(\tau_1 = 0.3\) and \(\tau_2 = 0.4\), the initial state \(x_1(t) = 2\) for \(t \in [-\tau_1, 0]\) and \(x_2(t) = -3\) for \(t \in [-\tau_2, 0]\); Case 2 with the same delay parameters as in Case 1 and the initial state \(x_1(t) = 5\) for \(t \in [-\tau_1, 0]\) and \(x_2(t) = 3\) for \(t \in [-\tau_2, 0]\), Case 3 with the delay parameters \(\tau_1 = \tau_2 = 0.3\) and the initial state \(x(t) = (-4.5, -1)^T\) for \(t \in [-0.3, 0]\), Case 4 with the same delay parameters as in Case 3 and the initial state \(x(t) = (-2, 6)^T\) for \(t \in [-0.3, 0]\). Fig. 3 depicts the time responses of state variables of the neural network.

**Example 2:** Consider another memristor-based recurrent neural network with time-delays:
\[\dot{x}(t) = -Dx(t) + A(x)f(x(t)) + B(x)g(x(t - \tau)) + s, \quad (24)\]
where
\[D = \begin{pmatrix} 19 & 0 \\ 0 & 15 \end{pmatrix}, \quad A(x) = \begin{pmatrix} 7 & a_{12} \\ a_{21} & -3 \end{pmatrix}, B(x) = \begin{pmatrix} -4 & b_{12} \\ b_{21} & -5 \end{pmatrix}, \quad a_{12} = \begin{cases} 
2.5, & \dot{x}_1 > \dot{x}_2 \\
-1, & \dot{x}_1 < \dot{x}_2
\end{cases}, \quad a_{21} = \begin{cases} 
3.5, & \dot{x}_2 > \dot{x}_1 \\
5, & \dot{x}_2 < \dot{x}_1
\end{cases}, \quad b_{12} = \begin{cases} 
4, & \dot{x}_1 > \dot{x}_2 \\
-3, & \dot{x}_1 < \dot{x}_2
\end{cases}, \quad b_{21} = \begin{cases} 
-2, & \dot{x}_2 > \dot{x}_1 \\
-3.5, & \dot{x}_2 < \dot{x}_1
\end{cases}.
\]
and

\[ f_i(x) = g_i(x) = \frac{1}{2}(|1+x| - |1-x|), \quad i = 1, 2, \]

are Lipschitz continuous with the Lipschitz constants \( \mu_i = \lambda_i = 1. \)

Note that

\[ |A|_{\text{max}} = \begin{pmatrix} 7 & 2.5 \\ 5 & 3 \end{pmatrix}, \quad |B|_{\text{max}} = \begin{pmatrix} 4 & 4 \\ 3.5 & 5 \end{pmatrix}, \]

and

\[ D - |A|_{\text{max}}K - |B|_{\text{max}}L = \begin{pmatrix} 8 & -6.5 \\ -8.5 & 7 \end{pmatrix}. \]

Since \( D - |A|_{\text{max}}K - |B|_{\text{max}}L \) is an M-matrix, according to Theorem 1, system (24) is globally uniformly asymptotically stable.

Let \( s = (-3, -6)^T. \) The following four cases are given: Case 1 with the delay parameters \( \tau_1 = 0.4 \) and \( \tau_2 = 0.6, \) the initial state \( x_1(t) = 5 \) for \( t \in [-\tau_1, 0] \) and \( x_2(t) = -2 \) for \( t \in [-\tau_2, 0]; \) Case 2 with the same delay parameters as in Case 1 and the initial state \( x_1(t) = -4 \) for \( t \in [-\tau_1, 0] \) and \( x_2(t) = -3 \) for \( t \in [-\tau_2, 0]; \) Case 3 with the delay parameters \( \tau_1 = \tau_2 = 0.6 \) and the initial state \( x(t) = (2, 3.5)^T \) for \( t \in [-0.6, 0]; \) Case 4 with the same delay parameters as in Case 3 and the initial state \( x(t) = (-1, 2.5)^T \) for \( t \in [-0.6, 0]. \) Fig. 4 depicts the time responses of state variables of the neural network for the above four cases.

V. CONCLUSIONS

In this paper, a piecewise-linear mathematical model of the memristor is first given to characterize its feature of pinched hysteresis and a recurrent neural network model with time delays based on this model is then proposed. Such a model is basically a state-dependent nonlinear switching dynamical system. To study the stability of a switching system, the conventional approach is the multiple Lyapunov function (MLF) method. The idea of MLF is rather simple. If for each subsystem we can find a Lyapunov function that satisfies certain conditions, then the whole system is stable. However, for the memristor-based network, since it consists of too many subsystems, it is too difficult to do so. In this paper, we use differential inclusion to avoid this difficulty. Under the framework of Filippov’s solution, we can turn to analyze the stability of a relevant differential inclusion which is upper semi-continuous and easier to handle. We obtain two sufficient conditions of the global uniform asymptotic stability of memristor-based recurrent neural networks with time-delays which are the generalization of those for conventional recurrent neural networks.
REFERENCES


[22] B. E. Paden and S. S. Sastry, “A calculus for computing Filippov’s differential inclusions with applications to the variable structure control