Structural Shape and Topology Optimization in a Level Set Based Implicit Moving Boundary Framework

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Abstract

In this paper we present a new framework to approach the problem of structural shape and topology optimization. We use a level set method with an implicit moving boundary model. As a boundary optimization problem, the structural boundary description is implicitly embedded in a scalar function as its “iso-surfaces.” Such level set models are flexible in handling complex topological changes and are concise in describing the boundary shape of the structure. Furthermore, by using a simple Hamilton-Jacobi convection equation, the movement of the implicit moving boundaries of the structure is driven by a transformation of the objective and the constraints into a speed function that defines the level set propagation. The result is a 3D structural optimization technique that demonstrates outstanding flexibility of handling topological changes, fidelity of boundary representation and degree of automation, comparing favorably with other methods based on explicit boundary variation or homogenization in the literature. We have developed a number of numerical techniques for an efficient and robust implementation of the proposed method. The method is tested with several examples of a linear elastic structure that are widely reported in the topology optimization literature.

Keywords

Structural optimization, topology optimization, shape optimization, boundary optimization, level set models, level set methods, implicit moving boundary

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1 Introduction

In this paper we address the problem of shape and topology optimization of a linearly elastic structure to meet a design objective and to satisfy certain constraints. The problem is formulated in an implicit moving boundary framework in that the design domain is described by the structural boundary which is embedded in a scalar function of a higher dimensionality. As a level set or an “iso-contour” of the embedding function (also called the level set function), the boundary is implicitly described without the need of an explicit representation. The optimization process is captured by a Hamilton-Jacobi type partial differential equation (PDE) that governs the dynamic movement of the embedding function and hence changes in the structural boundary in accordance with the design objective and the constraints. While the shape and connectivity (i.e., topology) of the boundary may undergo drastic changes, the level set function remains to be simple in its topology. Therefore, by a direct and efficient computation in the embedding space, the design boundaries can be tracked to a required level of accuracy yielding an optimal structure in both shape and topology. The level set models are referred to as an implicit moving boundary (IMB) framework and they can easily represent complex boundaries that can form holes, split into multiple pieces, or merge with others to form a single one. Based on the concept of propagation of the level set interface, an optimization algorithm is derived from the shape sensitivity and the variations of the level-set embedded boundary.

Boundary-based shape optimization has been a major method for structural design [20, 30]. In essence, the design domain is directly represented by its boundary and a set of design variables directly control the exterior and interior boundary shapes, for example, through the control points of B-splines. Based on a boundary shape sensitivity analysis, necessary boundary variations for the optimality conditions would provide the foundation of an optimization technique [30]. It is a direct approach and it is concise in the sense that the geometric boundary of the structure is expressed explicitly. Therefore, it generally allows more explicit representation of any features to be incorporated in the design. A major limitation of an explicit boundary representation, however, is that the connectivity of the boundary, or the topology of the structure, is fixed. A conventional boundary-based approach is not capable of handling topological changes for structural optimization [4, 20].
Perhaps as a major motivation to overcome the fixed-topology limitation, various other techniques and approaches have been developed during the past decade. As the state-of-the-art, homogenization-based methods have become the main approach to structural optimization [1-7], in which a material model with micro-scale voids is introduced and the topology optimization problem is defined by seeking the optimal porosity of such a porous medium using one of the optimality criteria. By transforming the difficult topology design problem into a relatively easier “sizing” problem, the homogenization technique is capable of producing internal holes without prior knowledge of their existence. That is, it offers a tool for simultaneous shape and topology optimization. A number of variations of the homogenization method have been investigated to deal with these issues by penalization of intermediate densities, especially the “solid isotropic material with penalization” (SIMP) approach for its conceptual and practical simplicity [4, 15, 21]. Material properties are assumed constant within each element used to discretize the design domain and the design variables are the element densities. The material properties are modeled to be proportional to the relative material density raised to some power. The power-law based approach to topology optimization has been widely applied to problems with multiple constraints, multiple physics and multiple materials [4, 10, 9, 21, 27, 28, 31].

However, the homogenization method may not yield the intended results for some objectives in the mathematical modeling of structural design. It often produces designs with infinitesimal pores in the materials that make the structure not feasible. Further, numerical instabilities may introduce “non-physical” artifacts in the results and make the designs sensitive to variations in the loading [4, 9, 21]. Numerical instability and computational complexity remain to be the major difficulties and are encountered in every realistic application. In our viewpoint, the root of these problems may well be related to the very reason for its success in other respects: its elimination of boundary description. By treating the optimization problem as a material distribution problem, the homogenization-based approach (or the SIMP method) has fundamentally changed the nature of the problem. While there exists no geometric boundaries in the problem domain, there is no boundary connectivity to deal with; thus, there are no topological changes in a fundamental sense. In the end, the designer must interpret the resulting material distribution and extract a boundary and topological description which is essential for obvious reasons [14]. These fundamental issues are still argued in the literature [21].

Another class of approach is essentially based on an evolutionary strategy called “evolutionary structural optimization” (ESO), focusing on local consequences but not on the
global optimum. It is typically computationally expensive since it has to rely on a “greedy-type” algorithm. A simple method of this class of optimization has been proposed by Xie and Steven [33] which is based on the concept of gradually removing material to achieve an optimal design. The method was developed for various problems of structural optimization including stress considerations, frequency optimization, and stiffness constraints. A similar approach called “reverse adaptivity” was proposed by Reynolds et al. [19] at which a fixed percentage of relatively under-stressed material is removed to find approximately fully stressed structures. Essentially, both evolutionary structural optimization and reverse adaptivity are homotopy methods based on so-called material “hard kills”. In reverse adaptivity finite element meshes near the boundary during the design procedure are refined to reduce computational cost or increase resolution.

Another approach is called “bubble method” proposed by Eschenauer and Schumacher [11, 12]. In the method, so-called characteristic functions of the stresses, strains and displacements are employed to determine the placements or insertion of holes of known shape at optimal positions in the structure, thus modifying the structural topology in a prescribed fashion. In such case, the design for a given topology is settled before its further changes.

Adopting the same principle of redesigning the structure based on the stress distribution in the current design, another approach was developed by Sethian and Wiegmann [23] with a focus on the resolution of the boundaries. The boundaries are allowed to move according to the stresses on the boundaries. A level set method is employed for tracking the motion of the structural boundaries under a speed function and in the presence of potential topological changes. An explicit jump immersed interface method is used for computing the solution of the elliptic problem in complex geometries without using meshes. The approach is also an evolutionary one. The principal idea is to remove material in regions of low stress and to add material in regions of high stress. A removal rate is established representing a percentage of the maximal initial stress below which material may be eliminated, and above which material should be added. The removal rate determines the closed stress contours along which new holes are cut and also the velocity of the boundary motion. The biggest benefit of this approach is that it is easier to add material (with some sub-grid resolution) at holes’ boundaries with high stress than on a triangulated finite element mesh. This approach seeks to improve design by making more efficient use of the material.
In our point of view, a boundary-based method with the capability of handling topology changes has the most promising potential. Boundary representations are always essential for design description and for design automation with CAD and CAE systems. A major contribution of our work presented in this paper is the capability to capture topological changes with an implicit moving boundary (IMB) representation based on the level set methods. In our approach presented here, the structural boundaries are viewed as moving during the optimization process – interior boundaries (or holes) may merge with each other or with the exterior boundary and new holes may be created. The shape and topology optimization is carried out as a moving front of the level sets driven by the dynamics of the implicit boundary under optimization conditions.

2 The Optimization Problem

In this paper we use a linear elastic structure to describe the problem of structural optimization. Conceptually, the approach presented here would apply to a general structure model. Let \( \Omega \subseteq \mathbb{R}^n \) \( (n = 2 \text{ or } 3) \) be an open and bounded set occupied by a linear isotropic elastic structure. The boundary of \( \Omega \) consists of three parts \( \Gamma = \partial \Omega = \Gamma_d \cup \Gamma_u \cup \Gamma_r \), with Dirichlet boundary conditions on \( \Gamma_u \) and Neumann boundary conditions on \( \Gamma_r \). It is assumed that \( \Gamma_d \) is traction free. The displacement field in \( \Omega \) is the unique solution of the linear elastic system

\[
- \text{div} \, \sigma(u) = 0 \quad \text{in} \quad \Omega \\
u = u_0 \quad \text{on} \quad \Gamma_u \\
\sigma(u)N = \tau \quad \text{on} \quad \Gamma_r
\]

where the strain tensor \( \varepsilon \) and the stress tensor \( \sigma \) at any point \( x \in \Omega \) are given in the usual form as

\[
\sigma_{ij}(u) = E_{ijkl} \varepsilon_{kl}(u) \\
\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

with \( E_{ijkl} \) to be the elasticity tensor, \( u_0 \) the prescribed displacement on \( \Gamma_u \), \( \tau \) the boundary traction force applied on \( \Gamma_r \), and \( N \) the outward normal to the boundary.

The general problem of structure optimization is specified as
Minimize  \( J(u) = \int_{\Omega} F(u) d\Omega \)

subject to:

\[
\int_{\Omega} E_{ijl} e_{ij}^f(u) e_{kl}^s(v) d\Omega = \int_{\Gamma} p v d\Omega + \int_{\Gamma} \tau v d\Gamma, \quad \text{for all } v \in U
\]  \( \tag{3} \)

\[
u = u_0 \quad \text{on } \Gamma_u
\]

\[
\int_{\Omega} d\Omega \leq V_{\text{max}}
\]

Here, the linear elastic equilibrium equation is written in a weak variation form, with \( U \) denoting the space of kinematically admissible displacement fields and \( p \) representing the applied body forces. The inequality describes the limit on the amount of material in terms of the maximum admissible volume \( V_{\text{max}} \) of the design domain. The goal of structural optimization is to minimize the objective function \( J(u) \) for a specific physical or geometric type described by \( F(u) \). This is a standard notion of structural optimization [4, 30].

### 3 Implicit Boundary Models via Level Sets

Shape optimization is a general approach to the problem at hand. It is based on an analysis of shape sensitivity in terms of variations of the structural boundary. Standard procedures are well documented in the literature for obtaining the set of necessary conditions to be satisfied by an optimal solution (see, for example, [30].) A key concept in such an analysis is the “speed function” \( V_N \) of the optimality condition associated with a small variation in the boundary shape in the normal direction \( N \). In general, it is necessary that

\[
V_N(x) = 0
\]  \( \tag{4} \)

everywhere on the design boundary \( \Gamma_d \) of the optimal structure. Physically, this indicates that the mutual energy form of the elastic structure reaches a constant value on \( \Gamma_d \) [30]. In most shape optimization applications, directly solving this optimality equation is not possible. One general technique of shape optimization is to solve the ordinary differential equation

\[
\frac{dx}{dt} = V_N(x)
\]  \( \tag{5} \)
with given initial boundary shape. The auxiliary variable \( t \) is denoted as the time-marching parameter. This is also known as the Lagrangian formulation of boundary propagation. When the steady state of this equation is achieved (i.e., \( \frac{dx}{dt} = 0 \)), the optimality condition is also achieved and, hence, an optimal shape of the structure is obtained. This is the well-known gradient descent method and there exist a large number of algorithms [20, 30].

In this conventional boundary-based method, the moving boundary is usually discretized with a set of design variables directly controlling the exterior and interior boundaries. The discrete design variables can be computed iteratively with an optimization algorithm and a finite element analysis. This often requires the construction of a new discrete model of the structure, or re-meshing, after each iteration. These issues have been extensively studied and there are well-established numerical methods and software systems for boundary shape design of structures [30]. However, all of these methods utilize an explicit boundary representation and the boundary changes can be accomplished only if the connectivity of the boundaries does not change. In other words, they all have a sever limitation that only a structure of a fixed topology can be optimized. For this reason, a boundary-based optimization has often been referred to as shape optimization. Topology changes in a structure means that a boundary can “split” into pieces to form multiple boundaries or “holes”. Conversely, several distinct boundaries may merge to make a single boundary. These changes provide the greatest challenge in a boundary-based approach to structural optimization.

As opposed to tracking the structural boundary with the Lagrangian formulation of (5), we suggest to use an implicit function \( \Phi(x) \) both to represent the boundary and to optimize it, as it was originally developed for curve and surface evolution [16, 24]. The change of the implicit function \( \Phi(x) \) is governed by the simple convection equation

\[
\frac{\partial \Phi(x, t)}{\partial t} + \nabla \Phi(x, t) \cdot V(x) = 0
\]

(6)

where \( V(x) \) defines the “velocity” of each point on the boundary. This is also known as an Eulerian formulation of the boundary propagation since the boundary is captured by the implicit function \( \Phi(x) \) here. The velocity field \( V(x) \) comes from an important concept of embedding the structural surface boundary as an iso-surface of the implicit function, \( \Phi : \mathbb{R}^n \mapsto \mathbb{R} \), such
that

$$\Gamma = \{ x : \Phi(x) = k \}$$

(7)

where \( k \) is the iso-value and is arbitrary, and \( x \) is a point in space on \( \Phi \). In other words, \( x \) is the set of points in \( \mathbb{R}^n \) that composes the \( k^{\text{th}} \) iso-surface of \( \Phi \). The embedding \( \Phi \) of \((n+1) \) dimension can be specified in any specific form, for example, as a regular sampling on a rectilinear grid. A process of structural optimization can be described by letting the level set function dynamically change in time. Thus, the dynamic model is expressed as

$$\Gamma(t) = \{ x(t) : \Phi(x(t), t) = k \}$$

(8)

By differentiating both sides of (8) with respect to time and applying the chain rule, we obtain the so-called Hamilton-Jacobi equation

$$\frac{\partial \Phi(x, t)}{\partial t} + \nabla \Phi(x, t) \frac{dx}{dt} = 0$$

(9)

This equation defines an initial value problem for the time dependent function \( \Phi \). In this level set model, \( dx/dt \) is the velocity vector for shape optimization, \( dx/dt = V(x) \). Thus, the optimal boundary is expressed as the solution of the partial differential equation on \( \Phi \):

$$\frac{\partial \Phi(x)}{\partial t} = -\nabla \Phi(x) \frac{dx}{dt} \equiv -\nabla \Phi(x) \cdot V(x)$$

(10)

Furthermore, since the local unit normal to the surface \( N \) is always in the same direction with the gradient of the implicit function \( \nabla \Phi \),

$$N = \frac{\nabla \Phi}{|\nabla \Phi|} \quad \text{(where } |\nabla \Phi| = \sqrt{\nabla \Phi \cdot \nabla \Phi})$$

(11)

the tangential components of \( V \) would vanish in the equation. Thus, equation (10) can be written as
This is known as the level set equation [17, 22, 24]. As in (5) the normal velocity \( V_N \) is related to the sensitivity of the shape to the boundary variation and is dependent of the objective of the optimization.

This formulation with level set models has two major theoretical and practical advantages over conventional explicit boundary models, especially in the context of topology optimization. First, level set models are topologically flexible. The scalar function \( \Phi \) is defined to always have a simple topology; complicated surface shapes are implicitly represented by the level sets of \( \Phi \). The boundary shape representation is as general as the underlying physical theory. More importantly, the representation does not rely on any kind of explicit parameterization, along with no direct specification of the topology of the structure. These capabilities would allow the boundary models to easily change the structural topology while undergoing optimization in that they can form holes, split to form multiple boundaries, or merge with other boundaries to form a single surface. There is no need to re-parameterize the model as it undergoes significant changes in shape, in contrast to any conventional boundary shape design [24]. Further, the models can incorporate a large number of degrees of freedom and a number of numerical techniques have been developed [17, 22, 24] to make the initial value problem of (12) computationally robust and efficient. In fact, in the general case of a three dimensional solid structure, the computational complexity can be made proportional to the surface area of the structure rather than the size of its volume. We shall describe the details of numerical computation of our proposed approach in the later sections.

Within this implicit boundary framework our original problem of structural optimization (3) is described in terms of the level set model as follows. We define a larger, fixed reference domain \( \overline{\Omega} \) such that it fully contains the structure being optimized \( \Omega \), i.e., \( \Omega \subset \overline{\Omega} \). As described in Eq. (8), the boundary surface \( \partial\Omega \) is implicitly defined as an iso-surface of the embedding \( \Phi(x) \) such that \( \Gamma = \{ x : x \in \overline{\Omega}, \Phi(x) = 0 \} \). Here we use the convention that \( k = 0 \). Furthermore, the local sign of \( \Phi(x) \) can be used to define the inside and outside regions of the boundary such that

\[
\frac{\partial \Phi(x)}{\partial t} = -V_N |\nabla \Phi(x)| \tag{12}
\]
These regions and the level set embedding of the model are shown in Fig. 1 for a two-dimensional structure. In this case, the boundary curves are embedded in a three-dimensional function $\Phi(x)$ with a fixed topology. The surface of the embedding function may move up and down on a fixed coordinate system without ever altering its topology. The structural boundary curves embedded on $\Phi(x)$ can undergo drastic topological changes. However, there is no need to directly track these structural topological changes. The boundary optimization is implemented with the motion of $\Phi(x)$, and the topological changes in the boundary are discovered when the corresponding level set is computed.

Figure 1. The design domains and the boundary embedding with a level set model.
With the level set models we can describe the optimal design problem in terms of the scalar function $\Phi$. It is more convenient to use the Heaviside function $H$ and the Dirac delta function $\delta$ defined as

$$
H(\Phi) = \begin{cases} 
1 & \text{if } \Phi \geq 0 \\
0 & \text{if } \Phi < 0
\end{cases} \quad \text{and} \quad \delta(\Phi) = \frac{dH}{d\Phi}
$$

(14)

Therefore, the optimization problem is now written as follows:

\[
\begin{align*}
\text{Minimize} & \quad J(u, \Phi) = \int_{\Omega} F(u)H(\Phi)d\Omega \\
\text{subject to:} & \quad a(u, v, \Phi) = L(v, \Phi) \quad \text{for all } v \in U \\
& \quad u = u_0 \text{ on } \Gamma_u \\
& \quad V(\Phi) = \int_{\Omega} H(\Phi)d\Omega \leq V_{\text{max}}
\end{align*}
\]

(15)

where

$$
a(u, v, \Phi) = \int_{\Omega} \epsilon_{ij} \epsilon_{ij}(u) H(\Phi)d\Omega \\
L(v, \Phi) = \int_{\Omega} pv H(\Phi)d\Omega + \int_{\Omega} \nabla \delta(\Phi) |\nabla \Phi| d\Omega
$$

(16)

are the energy bilinear form and the load linear form respectively, and $V(\Phi)$ defines the volume of the structure.

4 Shape Sensitivity and Optimality Conditions

With the formulation of (15) we are now ready to derive the necessary optimality conditions for the construction of an optimization procedure. The principal guideline for the optimization process is to move the design boundary represented by the level set model according to its shape sensitivity with respect of the motion of the embedding function $\Phi$ as shown in Fig. 1(b). The key development of our application of the level set methods here is to find an appropriate “speed function” $V_\nu$ in (12) such that it will drive the design boundary into the optimum shape based on the given objective function and the constraint. The speed function must be expressed in terms of the level set function $\Phi$ and must be linked to the derivative of the objective function.
with respect to the level set variation. A highlight of our approach presented here is to bridge the well-established methods of shape sensitivity analysis [30] with the powerful methods of level sets [17, 24] to fulfill our goal of general structural optimization within the implicit boundary framework.

Using the standard Lagrangian multiplier method, we construct another objective function \( J(u, \Phi) \) and obtain a completely equivalent problem to the original optimization problem (15) as follows:

\[
\begin{align*}
\text{Minimize } \tilde{J}(u, \Phi) &= J(u, \Phi) + \lambda_c \cdot (V(\Phi) - V_{\text{max}}) \\
\text{subject to } & : \\
\Phi_0(u, v, \Phi), u|_{\text{bdy}} &= u_0 \text{ for all } v \in U \\
\lambda_c \cdot (V(\Phi) - V_{\text{max}}) &= 0 \\
\lambda_c &\geq 0
\end{align*}
\]

Here \( \lambda_c \) is the Lagrange multiplier and the last two constraints define a complementarity condition: When the inequality of \( V(\Phi) < V_{\text{max}} \) is true, then \( \lambda_c = 0 \); Otherwise when \( V(\Phi) = V_{\text{max}} \), \( \lambda_c > 0 \). In order to derive the shape sensitivity we follow the well-known approach of Murat and Simon (see, e.g., [30]). Thus, we define a perturbation of the optimal domain \( \Omega^0 \) as

\[
\Omega = (I + \psi)\Omega^0
\]

with \( \psi \) representing the perturbation vector field. The shape derivative of \( \tilde{J}(u, \Phi) \) at \( \Omega^0 \) is then defined as the Frechet derivative. This is a well-defined notion and it is derived in full details in [30] with

\[
\left\langle \frac{dJ(u, \Phi)}{d\Phi}, \psi \right\rangle = \int_{\Omega} \delta(\Phi)(\beta(u, w, \Phi) + \lambda_c) \psi d\Omega + \int_{\partial\Omega} \frac{\delta(\Phi)}{|\nabla \Phi|} \frac{\partial \Phi}{\partial N} \psi d\Gamma
\]

where \( w \) represents the adjoint displacement in the conjugate equation.
\[ a(v, w, \Phi) = \langle J_u (u, \Phi), v \rangle = \left[ \frac{\partial F(u)}{\partial u} v H(\Phi) d\Omega \right]_{\partial \Gamma}, \quad w|_{\partial \Gamma} = 0, \quad \forall v \in U \]  

and

\[ \beta(u, w, \Phi) = F(u) + p \nu - \pi \nu \kappa - E_{ijkl} \epsilon_{ij}(u) \epsilon_{kl}(w) \]  

with \( \kappa = \nabla \cdot (\nabla \Phi / \| \nabla \Phi \|) \) being the mean curvature of the level set surface. Thus, the Kuhn-Tucker condition of the optimal solution becomes

\[
\begin{align*}
\beta(u, w, \Phi) + \lambda_+ |_{\Omega^c} &= 0 \\
\lambda_+ \cdot (V(\Phi) - V_{\text{max}}) &= 0 \\
\lambda_+ &\geq 0
\end{align*}
\]  

Further, we then can construct the speed function \( V_N(x) \) from the following equations

\[
\begin{align*}
V_N(x) &= -\beta(u, w, \Phi) - \lambda_+ \\
\lambda_+ &= -\int_{\partial \Omega} \beta(u, w, \Phi) \, d\Gamma / \int_{\partial \Omega} d\Gamma
\end{align*}
\]  

This speed function \( V_N(x) \) essentially represents a non-local version of the exact shape sensitivity (19). It serves in the Hamilton-Jacobi equation for a gradient descent solution for the structural optimization:

\[
\frac{\partial \Phi}{\partial t} = V_n |\nabla \Phi| \quad \text{and} \quad \frac{\partial \Phi}{\partial N} |_{\partial \Omega} = 0
\]  

The reader is referred to [32] for a detailed proof of this derivation. Finally, we can describe our optimization algorithm as an iterative process as follows:

**Main Algorithm:**

**Step 1:** Initialize the level set function \( \Phi(x, 0) \) at \( t = 0 \), corresponding to an initial design \( \Omega \) in terms of its boundary \( \Gamma \).
Step 2: Compute the displacement field $u$ and the adjoint displacement field $w$ through the linear elastic system.

Step 3: Calculate the “speed function” $V_N(x)$ for $x$ on surface $\Phi(x)$ along the normal direction $N(x)$.

Step 4: Solve the level set equation (24) to update the embedding function $\Phi(x,t)$.

Step 5: Check if a termination condition is satisfied. If the condition is met, then a convergent solution is found. Otherwise, repeat Steps 2 through 5 until convergence. The termination condition is defined as

$$\int_{\Omega} |V_N(x)\delta(\Phi)|\nabla\Phi|d\Omega \leq \gamma$$

where $\gamma$ is a specified error limit.

5 Numerical Computation

There are a number of computational issues that are important to the proposed level set method. At the first glance, the technique seems to be expensive. By embedding the structural boundary as the level set of a higher dimensional function, a boundary curve of a two dimensional problem is transformed into a surface, while a boundary surface of a general 3D solid has to be treated as a volumetric object. However, the level set embedding is only defined at the particular zero set $\Phi(x,t)=0$. This fact has been exploited to develop highly efficient algorithms which reduce the computational complexity back to the physical level of the structural boundary [8, 24]. Further, a set of highly accurate and robust numerical algorithms have been developed for a discrete solution of the PDE of (24) [16-18]. In this section, we briefly describe some of the key aspects in the numerical implementation of the level set method. More importantly, we present several developments to further improve the computational speed, numerical accuracy and reliability.

5.1 Up-Wind Computation Schemes

The discrete solution to the Hamilton-Jacobi equation (24) is computed using finite differences over discrete time steps and on a discrete grid over the level set function. A highly robust and accurate computational method was developed by Osher and Sethian [16] to address
the problem of overshooting. Based on the notion of weak solutions and entropy limits, a so-called “up-wind scheme” is proposed to solve (24) with the following update equation

\[ \phi_{ijk}^{n+1} = \phi_{ijk}^n - \Delta t [\max(N_{ijk}, 0)\nabla^+ + \min(N_{ijk}, 0)\nabla^-] \]  

(25)

with

\[ \nabla^+ = \begin{bmatrix} \max(D_{ijk}^{-x}, 0)^2 + \min(D_{ijk}^{+x}, 0)^2 + \\ \max(D_{ijk}^{-y}, 0)^2 + \min(D_{ijk}^{+y}, 0)^2 + \\ \max(D_{ijk}^{-z}, 0)^2 + \min(D_{ijk}^{+z}, 0)^2 \end{bmatrix} \]

\[ \nabla^- = \begin{bmatrix} \max(D_{ijk}^{+x}, 0)^2 + \min(D_{ijk}^{-x}, 0)^2 + \\ \max(D_{ijk}^{+y}, 0)^2 + \min(D_{ijk}^{-y}, 0)^2 + \\ \max(D_{ijk}^{+z}, 0)^2 + \min(D_{ijk}^{-z}, 0)^2 \end{bmatrix} \]

Here, \( \Delta t \) is the time step, and \( D_{ijk}^{\pm x}, D_{ijk}^{\pm y} \) and \( D_{ijk}^{\pm z} \) are the respective forward and back difference operators in the three dimensions of \( x \in \mathbb{R}^3 \) separately for a general 3D solid. In addition, the time step \( \Delta t \) must be limited to ensure the stability of the up-wind scheme (25). The Courant-Friedrichs-Lewy (CFL) condition requires \( \Delta t \) to satisfy

\[ \Delta t \max |V_{N_{ijk}}| \leq \Delta_{\text{min}} \]  

(26)

where \( \Delta_{\text{min}} = \min(\Delta x, \Delta y, \Delta z) \) stands for the minimum grid space among the three dimensions [16]. Furthermore, in order to obtain highly accurate numerical results, the level set function \( \Phi(x, t) \) is often initialized as the signed distance function and to satisfy the Eikonal equation

\[ |\nabla \Phi(x, t)| = 1 \]  

(27)

In the numerical implementation, functions \( \delta(\Phi) \) and \( H(\Phi) \) have to be approximated with a first order accurate smoothed version such as defined in [17, 18]. We use the following version [32]
where $\xi$ is a parameter of choice to determine the size of the bandwidth of numerical smoothing. We found that a good value of $\xi$ is between $0.5 - 0.7$ times of the minimum grid width $\Delta_{\text{min}}$ when the Eikonal equation (27) is maintained.

The first order scheme is well known for its numerical stability. However, it is highly diffusive. It can be made higher order through a total variation diminishing (TVD) Runge-Kutta scheme [25]. In [24], a second order method in time is given as

$$
\begin{cases}
\phi_{\text{up}}^{n+1} = \phi_{\text{up}}^{n} - \Delta t \left[ \max(V_{N_{\text{up}}},0)\nabla^{+} + \min(V_{N_{\text{up}}},0)\nabla^{-} \right]
\end{cases}
\begin{cases}
\phi_{\text{down}}^{n+1} = \phi_{\text{down}}^{n} - \frac{\Delta t}{2} \left[ \max(V_{N_{\text{down}}},0)\nabla^{+} + \min(V_{N_{\text{down}}},0)\nabla^{-} \right]
\end{cases}
$$

where the second order space quantities $\nabla^{+}$ and $\nabla^{-}$ for discrete approximation are constructed with an essentially non-oscillatory (ENO) interpolation as fully described in [24, 26]. Here, we shall omit their details for brevity. We have implemented this so-called “high resolution” scheme and found that it is indeed more accurate than the first order scheme as claimed in the literature. In general, the linear elastic equation (3) may be solved by a finite element method.

### 5.2 Local Schemes of Level Set Computation

The up-wind solutions produce the motion of level set models over the entire range of the embedding, i.e., for all values of $\Phi$ in (24). Since the optimum structural boundary is defined to be a single model, i.e., at $k = 0$, the calculation of solutions over the entire range of iso-values is unnecessary. This forms the basis for “narrow-band” schemes that solve (24) in a narrow band of the grid nodes that surround the level set of interest [23, 24]. While the up-wind scheme makes the level set method numerically robust, the narrow-band scheme makes its computational complexity proportional to the boundary area of the structure being optimized.
rather than the size of the volume in which it is embedded. We reported the use of the narrow-band scheme in an earlier implementation of the level set method in [32].

Another efficient method has been developed in [18] by making the embedding function $\Phi$ as a distance function. Then, while the function $\Phi$ is maintained to be a signed distance function, a local computation of the level set requires update only those points where $\Phi = 0$. This local computation scheme is even simpler and more efficient. It has been shown that this method has a formal complexity of $O(N)$ in the 2D case and $O(N^2)$ in the 3D solid case, where $N$ is the size of the spatial grid in each direction of the level set [18]. In other words, the complexity of the level set model computation remains at the level of its physical dimension, not of the higher dimension of its embedding function. This advantage makes the local level set method practically attracting.

5.3 Velocity Extension and Re-Initialization of Level Set Function

In the level set formulation, we need the normal velocity $V_N$ in a neighborhood of the design boundary or the zero level set $\Gamma(t)$. As suggested in [24], the most natural way to extend $V_N$ off the design boundary is to let the velocity $V_N$ be constant along the curve normal to $\Gamma(t)$ such that

$$\nabla V_n \cdot \nabla \Phi = 0$$  \hspace{1cm} (30)

This leads to the following hyperbolic partial differential equation

$$\frac{\partial V_N}{\partial t} + S(\Phi) \frac{\nabla \Phi}{|\nabla \Phi|} \cdot \nabla V_N = 0$$  \hspace{1cm} (31)

where $S(\Phi)$ is the signature function of $\Phi$ defined as

$$S(\Phi) = \begin{cases} 
-1 & \text{if } \Phi < 0 \\
0 & \text{if } \Phi = 0 \\
+1 & \text{if } \Phi < 0 
\end{cases}$$  \hspace{1cm} (32)
In order to increase the regularity, \( S(\Phi) \) may be approximated by \( \Phi / \sqrt{\Phi^2 + \Delta_{\text{min}}^2} \). Accurate and robust numerical schemes, such as the first order upwind method, exist to compute discrete solutions to the partial differential equation of velocity extensions [18]. For simplicity of the presentation, the reader is referred to [17, 18] for detailed formulae.

Another important consideration in the local computation of zero level sets is re-initialization of the level set function [24]. In most cases, it is impossible to prevent \( \Phi(x,t) \) from deviating away from a signed distance function. Flat or steep regions may develop as the boundary moves, rendering computation of the normal vector, normal velocity and curvature at the places inaccurate. This would result in inaccurate design boundaries. For numerical reasons, the level set function need to be resurrected to be close to a signed distance function from time to time [18, 24]. We use another PDE-based method for this purpose by solving the following Hamilton-Jacobi equation to its steady state,

\[
\frac{\partial \Phi}{\partial t} = S(\Phi_0)(1 - |\nabla \Phi|)
\]

which results in the desired signed distance function of \( S(\cdot) \) [24]. This approach allows us to avoid finding the design boundary explicitly. As reported in the literature, again a first order discrete time upwind scheme with a second order essentially non-oscillatory (ENO) discrete space scheme would yield good results [17, 26].

### 5.4 Curvature Discretization

In the numerical implementation of a three dimension case, we need to approximate the normal \( N \) and the mean curvature \( \kappa = \nabla \cdot N \) of the surface boundary:

\[
N = \frac{\nabla \Phi}{|\nabla \Phi|} = \begin{bmatrix}
\Phi_x / \sqrt{\Phi_x^2 + \Phi_y^2 + \Phi_z^2} \\
\Phi_y / \sqrt{\Phi_x^2 + \Phi_y^2 + \Phi_z^2} \\
\Phi_z / \sqrt{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}
\end{bmatrix}
\]

\[
\kappa = \frac{(\Phi_y^2 + \Phi_z^2)\Phi_{xx} + (\Phi_x^2 + \Phi_z^2)\Phi_{yy} + (\Phi_x^2 + \Phi_y^2)\Phi_{zz} - 2\Phi_x\Phi_y\Phi_{xy} - 2\Phi_x\Phi_z\Phi_{xz} - 2\Phi_z\Phi_y\Phi_{yz}}{(\Phi_x^2 + \Phi_y^2 + \Phi_z^2)^{1.5}}
\]
We compute the value of $\kappa$ at grid points neighboring the zero level set and then interpolate its value on the design boundary whenever it is needed. In the discrete computation, $\kappa, \Phi_x, \Phi_y, \Phi_z, \Phi_{xx}, \Phi_{yy}$, and $\Phi_{zz}$ are all discretized by central difference, given as

$$
(\phi_{xy})_{jk} = \frac{\phi_{r+1,j,k} - \phi_{r+1,j-1,k} - \phi_{r-1,j+1,k} + \phi_{r-1,j-1,k}}{4\Delta x\Delta y} \quad (35)
$$

$$
(\phi_{xz})_{jk} = \frac{\phi_{r+1,j,k+1} - \phi_{r+1,j,k-1} - \phi_{r-1,j,k+1} + \phi_{r-1,j,k-1}}{4\Delta y\Delta z} \quad (36)
$$

$$
(\phi_{yz})_{jk} = \frac{\phi_{r+1,j+1,k} - \phi_{r+1,j-1,k} - \phi_{r-1,j+1,k} + \phi_{r-1,j-1,k}}{4\Delta z\Delta x}
$$

In order to enhance the robustness of the algorithm, we further confine the scope of the mean curvature as follows

$$
\kappa_{ijk} = \min(\Delta_{\min}, \max(\kappa_{ijk}, -\Delta_{\min})) \quad (37)
$$

This means that the minimum approximate value of the level set curvature is stipulated as the minimum grid width.

### 5.5 Nonlinear Mapping of the Moving Velocity

It is well known that the gradient descent method is not particularly efficient in the family of optimization methods. A significant improvement is to modify the gradient descent direction by a positive definite matrix approximating the Hessian matrix, namely, the conjugate gradient method. As for the infinite dimensional problem (15), it is difficult to construct such a similar function. Here we present a heuristic method aimed to improve the descent direction according the problem characteristics. The basic idea is to increase the difference of the velocity along the moving boundary while keeping the objective function descent. As a result, the speed of convergence of the PDE based optimization will be increased.

This heuristic idea requires us to define a nonlinear mapping function $f(\cdot)$ which satisfies the following conditions
a. \( f \) is continuous and closed in the tangential space \( T \) of the active constraint, 
\[
f(r) \in C^0 \text{ and } f(r) \in T \text{ for } \forall r \in T.
\]
b. \( f \) is an odd function, \( f(-r) = -f(r) \).
c. \( f \) is a non-decreasing function, \( f'(r) \geq 0 \).

It is not trivial to find a function satisfying these conditions. Here, we use the following procedure. First, we choose a nonlinear function \( F(r) \) which satisfies conditions (b) and (c). Then, let \( F(r) \) project on the tangential space \( T \) of the active constraint. Thus, a mapping function is constructed as follows

\[
f(V_n) = F(V_n) - \mu \quad \text{for} \quad V_n \in T
\]

where \( \mu \) is the average of the velocity function \( F(V_n) \) along the entire design boundary \( \Gamma \). As derived in [32] in detail, it is given as

\[
\mu = \int_{\partial\Omega} F(V_n) d\Gamma / \int_{\partial\Omega} d\Gamma
\]

With this nonlinear mapping function, the Hamilton-Jacobi equation (24) can be modified as

\[
\frac{\partial \Phi}{\partial t} = f(V_n) |\nabla \Phi| \\
\left. \frac{\partial \Phi}{\partial N} \right|_r = 0
\]

It can also be shown that the PDE generates a gradient descent solution to the problem of optimization (17) as the original PDE (24) such that \( dJ(u, \Phi)/dt \leq 0 \). A proof essentially follows the proof for the original PDE (24) as presented in our earlier work [32]. We shall omit the details here. It is also trivial to show that \( f(r) \) is an odd function due to the fact that \( F(r) \) satisfies condition (b). However, function \( f(r) \) constructed by using (30) is not assured to fulfill condition (c). When this happens, the nonlinear mapping of the velocity function may have a negative effect on the speed of convergence.
In our numerical implementation, we have used the following function $F(r)$ as illustrated in Fig. 2,

$$F(r) = r\left(\frac{1-\alpha}{2} + \frac{1+\alpha}{2}|r]\right)$$

where the $\alpha$ is a constant. In our numerical experience with examples to be presented in next section, we have found that this nonlinear mapping improves the computing efficiency significantly (by 2 – 3 times) compared to the direct gradient descent method.

![Figure 2. A nonlinear mapping function for the velocity field function.](image)

5.6 Regularization of the Level Sets

During the course of shape optimization with the level set models, it is possible that the boundary may not able to maintain certain level of smoothness due to numerical errors of discrete solutions. It is highly desirable that the irregularities are removed to enhance the fidelity of the level sets, while the meaningful discontinuities in the boundary representing topological changes remain to be kept. This is similar to the problem of “denoising” in image processing [22, 24]. This function can be easily incorporated into the implicit boundary framework, again due to the flexibility of the level set methods.
The regularization problem is defined as a variational problem [22]. We introduce a weighted length or area of the level set, namely,

$$E_s(\Phi) = \int_I I(x)d\Gamma = \int_I I(x)\delta(\Phi)\|\nabla \Phi\|d\Omega$$  \hspace{1cm} (42)

where $I(x) > 0$ corresponds to a Riemann metric. Thus the objective function of the optimization problem (15) is modified as

$$\tilde{J}(u, \Phi) = \int_I F(u)H(\Phi)d\Omega + E_s(\Phi)$$  \hspace{1cm} (43)

Again, taking the Fréchet derivative of this objective function with respect to $\Phi$ as in the original case, we obtain the gradient of the new objective function (43) at the zero level sets as follows

$$\beta(u, w, \Phi) = \beta(u, w, \Phi) - \nabla \left( I(x) \frac{\nabla \Phi}{\|\nabla \Phi\|} \right) = \beta(u, w, \Phi) - I(x)\nabla \left( \frac{\nabla \Phi}{\|\nabla \Phi\|} \right)$$  \hspace{1cm} (44)

This corresponds the total variation including regularization. The reason to use this variation form is straightforward. Here, the term $\nabla \cdot \left( \frac{\nabla \Phi}{\|\nabla \Phi\|} \right)$ is the mean curvature $\kappa$ of the level set. It is well known that a surface moving in its normal direction with the mean curvature as the velocity, also called the mean curvature flow, converges to the minimal surface. The mean curvature flow is also interpreted in the literature as an anisotropic diffusion [22], and it diffuses only in the tangential direction of the surface. Therefore, the regularization term in (42) plays a role in fairing the level sets only without any effect on their normal motion.

The regularization variation of (42) may be replaced by the so-called weighted total variation energy [17, 22]

$$E_{\gamma\nu}(\Phi) = \int_\Omega I(x)\|\nabla \Phi\|d\Omega$$  \hspace{1cm} (45)
with \( I(x) \) being regarded as the weighting coefficients. By deriving the Frenchet derivative for 
\( E_{TV}(\Phi) \) with respect to \( \Phi \), we will obtain the same gradient of the objective function as (44). Thus the regularization developed here can also be regarded as for reducing the total variation of the level sets.

The geometric metric \( I(x) \) is chosen according to the following criteria. (1) The regularization term in (44) should not have any significant influence on the process of optimization defined by the velocity function \( V_A(x) \). (2) As the level set moves to approach the optimum, the regularization must be enhanced gradually so as to obtain a smoothing or fairing of the structure boundary. In this study, we use the geometric metric as follows

\[
I(x) = \frac{c_1}{1 + c_2 V_N^2(x)}
\]  

(46)

where \( c_1 \) and \( c_2 \) are two positive constants used to shape the weighting function as shown in Fig. 3.

![Figure 3. Weighting function for fairing and smoothing the level sets.](image)

5.7 *Recovery of the Level Sets*
In the level-set based framework presented here, the geometric boundary of the structure under optimization is implicitly described as the zero level set of $\Phi(x,t) = 0$. There is no need to explicitly recover the boundary until the end of the optimization. There exist many techniques in most of the popular scientific software systems to compute iso-curves and iso-surfaces, essentially 2D and 3D level sets. For example, it is often convenient to describe the embedding function $\Phi$ as a rectangular sampling on a rectilinear grid of $x$ over $\Omega$ [24]. Then, the well-known marching-cubes technique in computer graphics can be directly applied to compute the zero level set of the optimal solution.

6 Numerical Examples

In this section we present several examples of structural optimization obtained with the proposed algorithm and implementation. The optimization problem of choice is the mean compliance problem that has been widely studied in the relevant literature (e.g., [4, 20]). The objective function of the problem is the strain energy of the structure with a material volume constraint, i.e.,

$$J(u) = \int_D E_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(u) d\Omega$$

(47)

For all examples, the material used is steel with a modulus of elasticity of 200 GPa and a Poisson’s ratio of $\nu = 0.3$. For clarity in presentation, the examples are in 2D under plane stress condition.

6.1 Michell-Type Structures

An Michell-type structure is first considered. A rectangular design domain of $L$ long and $H$ high with a ratio of $L:H = 12:6$ is loaded vertically at the center point of its bottom with $P = 30N$ as shown in Fig. 4. The left bottom corner of the beam is fixed, while it is simply supported at the right bottom corner. The volume ratio of 0.3 is considered. The initial design and some intermediate and the final optimization results are shown in Fig. 4. The final optimum solution is nearly identical to what other researchers have obtained using a homogenization based method (see [4, 21]). A mesh of $62 \times 122$ quadrilateral elements are used for the finite element analysis, and the numerical width $\xi$ for the approximate Heaviside
function is chosen to be 0.7 times of the grid width.

In the example, 60 holes are uniformly distributed in the initial design as shown, representing an original perforated structure. During the process of optimization, many of these holes would merge yielding “Swiss-cheese” structures during iteration. Many of the “Swiss-cheese” holes are gradually “gobbled up” by the level set processing. In the intermediate steps, some pieces of the structure may also break off to become separated. However, they are eventually evolved to disappear, since they are physically meaningless. This illustrates the flexibility and the power of the level set model to handle drastic topological changes. The convergence of the optimization process is shown with the changes of the objective function and the structure volume over iteration in Fig. 5.
Figure 4. A Michell type structure with fixed-simple supports. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
Figure 5. The changes in the mean compliance and the volume of the structure over iteration for example of Figure 4.

The second example is similar to the first example, except that the right bottom corner support is also fixed and its volume ratio is 0.2. The initial design and some intermediate and the final optimization results are shown in Fig. 6 and the changes of the objective function and the structure volume over the iteration is shown in Fig. 7.
Figure 6. A Michell type structure with fixed-fixed supports. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
6.2 Michell-Type Structures with Multiple Loads

An Michell-type structure is now considered with multiple loads at its bottom as shown in Fig. 8. The volume ratio is 0.3. Again, the mesh of $62 \times 122$ quadrilateral elements are used for FEM analysis. In Fig. 8, the structure has a fixed and a simple support at the bottom corners with $P_1 = 30N$ and $P_2 = 15N$. The initial design and some intermediate and the final optimization results are shown. Changes in the mean compliance and the body volume during the iterations of optimization are shown in Fig. 9.
Figure 8. A Michell type structure with fixed-simple supports and multiple loads $P_1 = 30N$ and $P_2 = 15N$. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
Figure 9. The changes in the mean compliance and the volume of the structure over iteration for example of Fig. 8.

Figs. 10 – 11 show the initial design and some intermediate and the final optimization results for three more cases of different values of loads of the example.
Figure 10. A Michell type structure with fixed-simple supports and multiple loads $P_1 = 30N$ and $P_2 = 30N$. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
Figure 11. Figure 8. A Michell type structure with fixed-simple supports and multiple loads $P_1 = 20N$ and $P_2 = 40N$. (a) Initial design. (b – g) Intermediate results. (h) Final solution.

In the third example of the Michell-type structure, it is similar to the example of Fig. 8 except that the supports are both fixed types. The volume ratio is 0.3, and the loads are $P_1 = 30N$ and $P_2 = 15N$. The optimization results are shown in Fig. 12.
Figure 12. A Michell type structure with fixed-fixed supports and multiple loads $P_1 = 30N$ and $P_2 = 15N$. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
6.3 MBB Beams

This example is said to be related to a problem of designing a floor panel of a passenger airplane in Germany known as Messerschmitt-Bolkow-Blohm (MBB) beams. The floor panel is loaded with a concentrated vertical force \( P = 80N \) at the center of the top edge. It is has a fixed support and a simple support at its bottom corners respectively. The design domain has a length to height ratio of 24:4. We use \( 56 \times 165 \) quadrilateral elements to model a half of the structure due to the geometric symmetry. The numerical width \( \xi \) of the Heaviside function is taken as 0.5 times of the grid width.

In Fig. 13 the volume ratio is specified to be 0.355, and the optimization results are shown. In Fig. 14, the volume ratio is slightly less as 0.35. As shown, the optimal design has a different topology of less number of holes when compared to the optimal design of Fig. 13 for the slightly higher volume ratio of 0.355.
Figure 13. A mid-point loaded beam (MBB beam) with fixed-simple supports and a volume ratio of 0.355. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
6.4 The Michell Trusses

The problem of Michell truss optimization is presented here. The design domain is shown in Fig. 15. An optimum structure is to be designed to transfer a vertical force to the circular fixed support. A force $P = 20N$ is applied to the middle of the right side. A mesh of $112 \times 82$ quadrilateral elements are used, the volume ratio is constrained to be 0.165, and the numerical width $\xi$ is specified to be 0.5 times of the minimum grid width. The optimization results of the example are shown in Fig. 15. Another example of the Michell truss is shown in
Fig. 16 with a different boundary condition and the same load of $P = 20N$. The volume ratio is constrained at 0.2. The FEM analysis uses a mesh of $86 \times 86$ quadrilateral elements.

Figure 15. The first Michell truss example. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
Figure 16. The second Michell truss example. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
6.5 **Cantilever Beams**

A cantilever beam of ratio 3.2:2 is loaded vertically at the center of its free end with a force of 80N as shown in Fig. 17. The volume ratio constraint of 0.25 is considered. The design domain is discretized using 22×34 quadrilateral elements. Two different values of $\xi$ are used for the smoothing of Heaviside function. The optimization results are shown in Figs. 17 and 18.

In Fig. 17, $\xi = 1.0$ is used. It means that the smoothing width for the Heaviside function is equal to the minimum grid width of the finite elements. In Fig. 18, a tighter width is used, $\xi = 0.75$. From the figures it is evident that both optimization processes arrive at the same final optimal solution, starting with the same initial design as shown. However, these two processes undertake different intermediate steps or optimization paths.

It is interesting to point out that in Fig. 18 with the tighter numerical width, an intermediate solution (Fig. 18(e)) is achieved. This intermediate design is very similar to optimal solutions that are often reported in the literature, especially those obtained with a homogenization (or SIMP) approach. In our process of optimization based on the level set models, the shape of the structure continues to evolve to approach to the final result as shown in Fig. 18(h). The final solution has a very different topology the intermediate solution with a slight improvement in the mean compliance. In this case, it shows that the problem of optimal topology may become very sensitive and multiple solutions may exist.
Figure 17. The first cantilever example with $\xi = 1.0$. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
Figure 18. The second cantilever example with $\xi = 0.75$. (a) Initial design. (b – g) Intermediate results. (h) Final solution.
6.6 Computational Time

Based on the numerical examples presented above, we further discuss the computational performance of the level set based method described above. There are two major computational tasks in the course of optimization. The first is the physical problem of linear elastostatic system of the solid structure. The underlying field equations are solved with the finite element method. The second task is the computation of the movements of the level sets to describe the changes in structural shape. For the examples that are examined, we found that it generally takes 80 – 250 iterations for an accurate optimal solution. The use of the nonlinear mapping of the velocity function can reduce the iteration number about 2.5 times when compared with a direct use of the velocity function as reported in our earlier work [32].

Furthermore, it is experienced that the FEM analysis takes the major portion of the total computational time and the level set updating uses only a small fraction. For the cantilever beam example, Table 1 tabulates the CPU time (in seconds) used in every iteration for the tasks of the level set updates and the FEM analysis for 4 different mesh sizes. It shows that the level set computation takes between 1.35% to 1.51% of the total computing time. Clearly, the level set method has shown its high efficiency. It also shows that the local level set computation method employed has a computing behavior of the linear complexity as predicted.

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<td>1.51%</td>
</tr>
</tbody>
</table>

Table 1 Comparison of the computing time for the cantilever beam examples.

7 Conclusions

We have presented an implicit moving boundary framework for structural shape and topology optimization based on the level set methods. As a boundary optimization problem, the structure is implicitly represented with a level set model that is embedded in a scalar function. The dynamics of the level set function is governed by a simple Hamilton-Jacobi convection equation. The movement of the implicit moving boundaries of the structure is driven by a transformation of the objective and the constraints into a speed function that defines the level set
propagation. The result is a 3D structural optimization technique that demonstrates outstanding flexibility of handling topological changes, fidelity of boundary representation and degree of automation, comparing favorably with other methods based on explicit boundary variation or homogenization in the literature.

We have developed a number of numerical techniques for an efficient and robust implementation of the proposed method. A local computation scheme is used to keep the computational complexity linear to the complexity of the physical boundary of the structure, while a second order discrete method based on the TVD Runge-Kutta scheme is used for accurate and stable numerical solution. We have developed a technique of nonlinear velocity mapping to substantially increase computational efficiency from the conventional gradient descent method for a faster convergence. The concept of regularization with an an-isotropic diffusion of mean curvature flow is utilized to maintain the boundary smoothness without sacrificing its fidelity to topology. The proposed approach is tested with several examples of mean compliance optimization of a linear elastic structure, as they have been widely analyzed in the literature. The approach can certainly be applied to other problems of structural optimization involving multi-physics and/or multi-domains.

The work presented in this paper is by no means complete. Our current algorithm is capable of creating new holes only from the Dirichlet boundary as reported in [32]. The creation process is typically slow. The level set models are at ease to handle topological changes of merging or cancellation of holes. Therefore, the algorithm described here is efficient to perform topology optimization if a number of holes exist in the initial design. An immediate need is to develop a technique for the creation of holes as they are needed. In fact, the concept of “topological derivatives” has been discussed in the literature recently [29], and it seems to have a close connection with the concept of the “characteristic function” of the bubble method [11, 13]. Thus, its application would add a significant “nucleation” capability to the level-set based implicit moving boundary framework. It is foreseeable that our proposed optimization system based on this framework would able to generate, split, merge, or diminish holes or cavities within the structure as well as to move the interior and exterior boundaries to ultimately achieve an optimal design. Our preliminary investigation in this direction is promising and we shall report our complete results separately.
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